Discussion Paper Series
No. 102

Department of Urban Engineering
University of Tokyo

Spatial relations among polygons: an exploratory analysis

Yukio Sadahiro
Department of Urban Engineering, University of Tokyo

Spatial relations among polygons: an exploratory analysis

Yukio Sadahiro
Department of Urban Engineering, University of Tokyo

Abstract

Polygons are fundamental spatial objects used in GIS that represent two-dimensional entities in the real world. They represent buildings, blocks, ponds, lakes, ethnicity and race distributions. These entities are often closely related with each other. For instance, the shape and location of buildings are restricted by streets and blocks. Ethnicity distribution is based on other cultural distributions such as those of race and religion. Since these relations among spatial objects have drawn much attention of geographers, analytical methods have been developed in geography and other related fields, from exploratory methods such as visual overlay of polygons to more sophisticated confirmatory ones. Standing in the same line as exploratory spatial analysis, this paper proposes a new method of analyzing the relations among polygons. The objective of analysis is to understand the global relation among polygons and to detect interesting and useful local patterns in the relations among polygons. A focus is on the hierarchical relation in topology of polygons, since the hierarchical system is an important concept in geography. Based on the topological relations between a pair of tessellations, network representations are proposed to represent the relations among more than three polygons. They allow us to classify polygons in terms of the closeness of their relations. The method is applied to analysis of the result of experiment in environmental psychology. Technical soundness of the method is discussed as well as empirical findings.

1. Introduction

Polygons are fundamental spatial objects used in GIS that represent two-dimensional entities in the real world. In geographical scale, polygons are used as computational models of artificial entities such as buildings and blocks, natural entities such as ponds and lakes, and even hypothetical or non-physical entities including administrative units, market areas, and ethnicity distributions. Even the region, a fundamental concept of geography, is often represented as a polygon in computer environment.

Spatial entities represented as polygons are often closely related with each other. The shape and location of buildings are restricted by streets and blocks. Land cover pattern is affected by topographic factors including soil and vegetation distributions. Ethnicity distribution is based on other cultural distributions such as race, religion, language, and occupation. Christaller's hierarchical system (Christaller, 1933) gives a theoretical model of the relations among administrative units and market areas.

Since these relations have drawn much attention of geographers, analytical methods have been developed in geography and other related fields. A naive method is to overlay the polygons on a single map and analyze the map visually at both global and local scales. Though it is an important first step of spatial analysis, it is not effective when numerous polygons overlap with each other. A more sophisticated approach is to build a spatial regression model (Anselin, 1988; Fotheringham *et al.*, 2000, 2002; Ward and Gleditsch, 2008; LeSage, 2009), where both dependent and independent variables are binary functions representing the existence or absence of polygons. This is a confirmatory approach suitable for testing research hypotheses on the cause and effect relationship among polygons. At an early stage of spatial analysis, however, such hypothesis has to be sought in an exploratory way. Quantitative as well as qualitative approaches are indispensable to find spatial patterns in the relations among polygons.

To answer this demand, this paper proposes a new exploratory method of analyzing the relations among polygons. The objective of analysis is to understand the global relation among polygons and to detect interesting and useful local patterns in the relations among polygons, in order to build research hypothesis for confirmatory spatial analysis. A focus is on the hierarchical relation in topology of polygons, since the hierarchical system is an important concept in geography.

Section 2 defines basic topological relations between a pair of polygons. Quantitative measures are also proposed to evaluate the closeness between polygons. Section 3 introduces two graph representations of the relations among polygons. Their definition and applications are discussed as well as computational procedures of constructing graph representations. Section 4 applies the proposed method to analysis of the result of an experiment in environmental psychology. The method is evaluated in terms of both technical soundness and empirical findings. Section 5 summarizes the conclusions with discussion for future research.

2. Topological relations between a pair of polygons

This section introduces basic topological relations between a pair of polygons, and proposes two quantitative measures to evaluate their closeness. Spatial relations have been widely and extensively discussed in computer science. Several definitions have been proposed in the literature based on point set topology.

Guting (1988) proposes five topological relations with a view to spatial operations: *equal*, *not equal*, *inside*, *outside*, and *intersects*. Egenhofer and Franzosa (1991) extends the definition by clearly distinguishing the boundary and inside of polygons. This yields eight topological relations: *equal*, *inside*, *contains*, *covers*, *coveredBy*, *overlap*, *meet*, and *disjoint*. Randell et al. (1992) also proposes an equivalent set of eight topological relations for qualitative spatial reasoning called the Region Connection Calculus (RCC): *equal*, *tangential proper part*, *tangential proper part inverse*, *non-tangential part*, *non-tangential part inverse*, *partially overlapping*, *externally connected*, and *disconnected* (Cohn et al., 1997). Further extensions have been done in the literature such as Egenhofer et al. (1994), Stock (1997), and Renz (2002).

The above definitions of topological relations were developed mainly for spatial reasoning and operation performed on a pair of polygons. This paper, on the other hand, focuses on the exploratory analysis of the relations among more than two polygons. Since numerous polygons are considered simultaneously, the set of topological relations should be as small as possible. We thus start with a basic set of relations defined between polygons.

Suppose a set of polygons $\Omega=\{P_i, i\in\mathfrak{M}=\{1, 2, ..., M\}\}$ in region S, where $\mathfrak{M}=\{1, 2, ..., M\}$. Each polygon is a closed set of points, which may be disconnected and have holes inside it.

Consider two polygons P_i and P_j in Ω, where $A(P_i)\geq A(P_j)$ ($A(P_i)$ gives the area of P_i). The polygons P_i and P_j are *equal* if they consist of the same set of points. Polygons are *hierarchical* if one contains the other in its inside. The larger polygon P_i is a *higher-level polygon* of the smaller one P_j while P_j is a *lower-level polygon* of P_i. When P_i and P_j partially overlap, they are called *overlapped*. The polygons are called *disjointed* if they do not share any point.

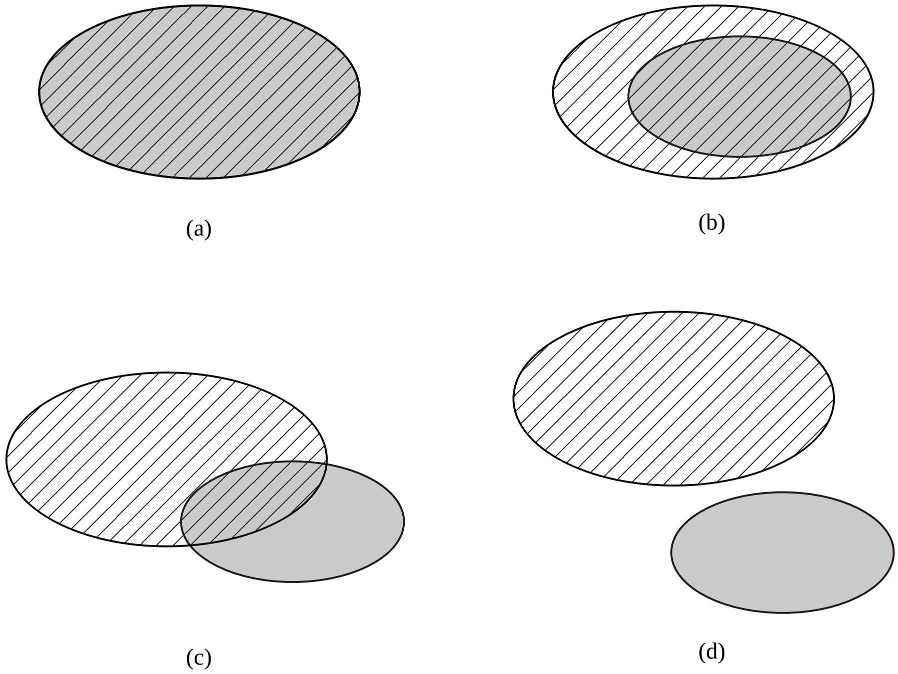


Figure 1 Relations between a pair of polygons. (a) Equal, (b) hierarchical, (c) overlapped, and (d) disjointed. These four relations are mutually exclusive.

To evaluate the closeness between P_i and P_j, we define two numerical measures as follows. *Size distance* between P_i and P_j is given by

$$D_S\left(P_i, P_j\right) = A\left(P_i\right) - A\left(P_j\right). \tag{1}$$

Hierarchy distance between P_i and P_j is the area of the relative complement of P_i in P_j:

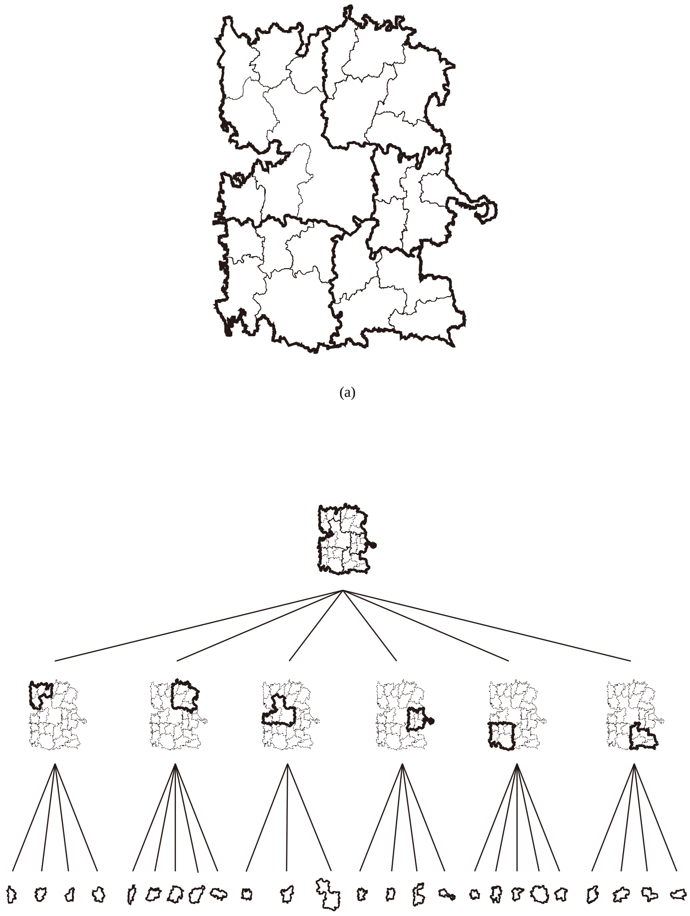
$$\begin{aligned} D_H\left(P_i, P_j\right) &= A\left(P_j \setminus P_i\right) \\ &= A\left(P_i \cup P_j\right) - A\left(P_i\right) . \\ &= A\left(P_j\right) - A\left(P_i \cap P_j\right) \end{aligned} \tag{2}$$

This measure evaluates the departure from complete hierarchy. If P_j and P_i are completely hierarchical, P_j is fully contained in P_i. The relative complement of P_i in P_j is empty and thus the distance becomes zero.

3. Graph representations of the relations among polygons

Using the hierarchical relation defined above, this section proposes graph representations of the relation among more than two polygons.

Spatial hierarchy is one of the most important concepts in geography since it is often found in the real world. A typical example is administrative systems where spatial units form a multi-level spatial hierarchy of polygons (Figure 2a). In such a case, hierarchical relation can be represented as a tree graph (Figure 2b).



(a)

(b)

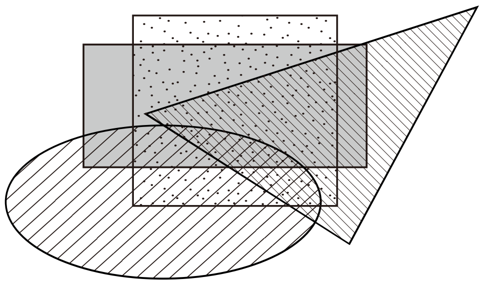
Figure 2 Hierarchical structure of administrative units. (a) Spatial configuration of country, states (bold lines), and towns (thin lines). (b) Tree representation of the hierarchical structure of administrative units. Upper, middle and lower rows indicate country, states and towns, respectively.

In the real world, however, polygons do not always show such a clear hierarchical relation. Even a slight overlap of polygons violates hierarchical relation. It is quite rare that all the polygons form a multi-level hierarchical system, especially when polygons represent cultural or natural territories.

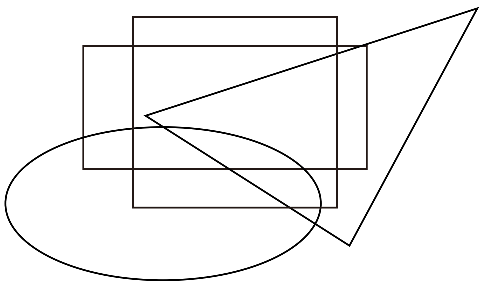
To treat a wider variety of situations, this paper proposes more flexible representations of the relations among polygons. Introducing the notion of lattice, we visualize the relations among polygons by graph representations (Anderson, 2002; Pemmaraju and Skiena, 2003).

3.1 Hasse diagram

Given a set of polygons $\Omega=\{P_i,\ i\in\mathfrak{M}\}$, we make the intersection of all the polygons by keeping the boundary lines. This often results in numerous fragmented polygons (Figure 3). The power set of the fragmented polygons, which is denoted by Λ, and Boolean operations $\{\cap, \cup\}$ form a lattice, where the least and greatest elements are $\varnothing$ and the union of all the polygons.



(a)



(b)

Figure 3 Intersection of four polygons. (a) Original polygons, (b) polygons generated by intersection.

A lattice as a poset (partially ordered set) can be visualized as a Hasse diagram (Birkhoff, 1979; Davey and Priestley, 2002). In the Hasse diagram, nodes and links represent polygons in Λ and topological relation between them, respectively (Figure 4).

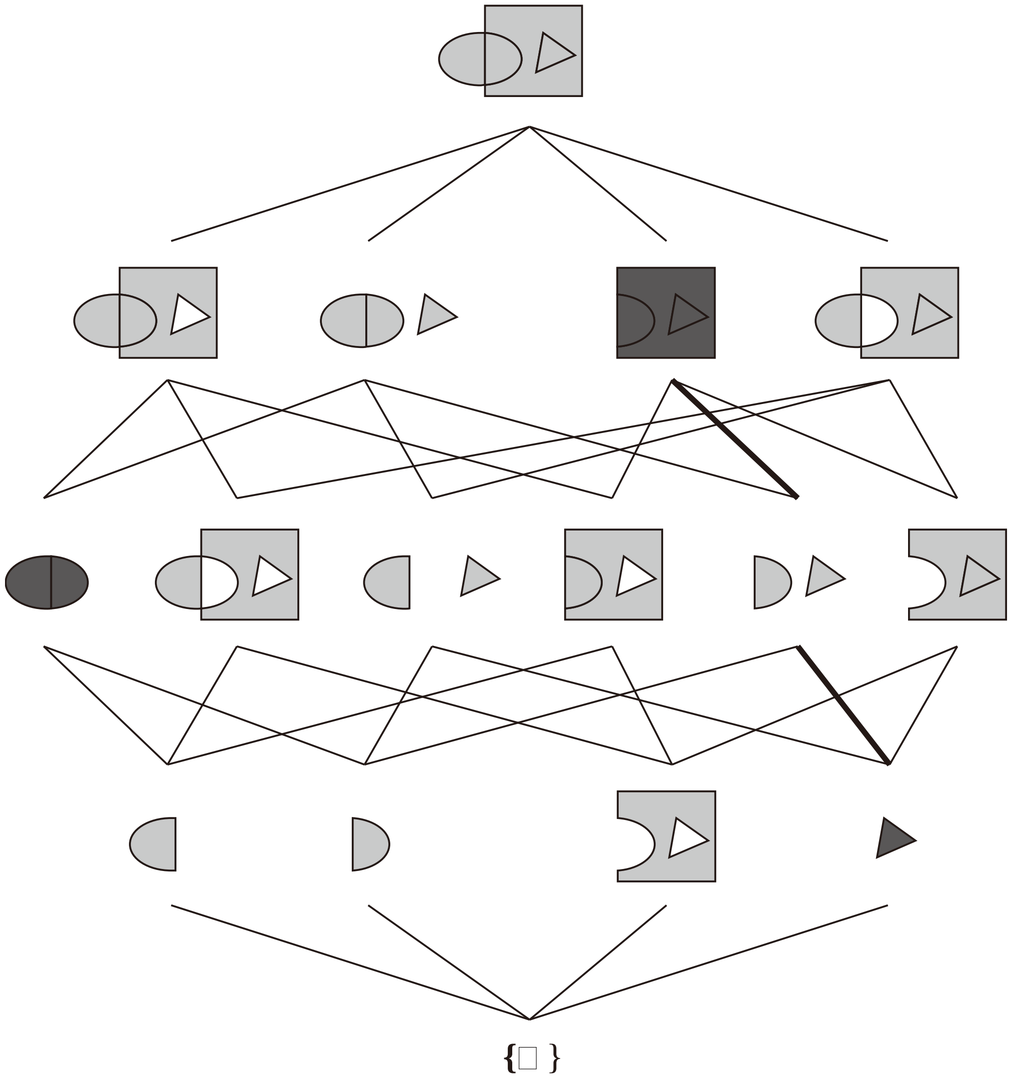


Figure 4 Hasse diagram of the power set of polygons generated from a rectangle, an ellipse and a triangle. Original polygons are indicated by darker gray. Tracing links downwards from the rectangle to the triangle (the trail is indicated by bold lines), we can confirm the hierarchical relation between them. From the ellipse and triangle, downward trails do not meet until the empty set at the bottom. This implies that they are disjointed. If the trails meet before reaching the bottom, they are overlapped.

By adding intermediate polygons generated by polygon overlay, Hasse diagram embeds all the original polygons as nodes in a graph. Consequently, it visualizes the relations among polygons even if they are not hierarchical. Hierarchical polygons are connected by either a single link or a set of unidirectional links.

Given two polygons in Λ, we can find their relation by tracing links downward from the polygons. The polygons are hierarchical if one polygon is reachable by tracing links downwards from the other. We can confirm the hierarchical relation between the rectangle and the triangle in Figure 4 since the triangle is reachable by tracing downwards the links indicated by the bold line.

If one polygon is not reachable from the other, they are either overlapped or disjointed. In this case, we go down along the links from the two polygons until the two trails meet. The polygon where the trails meet is the intersection of the two polygons. If the trails do not meet until the least element ∅, the polygons are disjointed. The ellipse and the triangle in Figure 4 are disjointed since the trails do not meet until reaching the bottom of Hasse triangle.

3.2 Quasi-Hasse diagram

Hasse diagram is a useful tool for visualizing the relation among polygons. However, it is not effective for a large set of polygons. Since polygons generated by the intersection of original ones rapidly increases in $O(2^M)$, calculation of their power set is practically impossible. Hasse diagram inevitably becomes too large and complicated, and consequently, it is quite difficult to understand the whole structure of the relations among original polygons.

To complement Hasse diagram, this paper proposes another graph representation what we call *quasi-Hasse diagram*. It is a degenerated subset of Hasse diagram whose vertical axis indicates the size of polygons. The quasi-Hasse diagram is constructed by the three steps described in the following three subsections.

3.2.1 Construction of intersection and union trees

Quasi-Hasse diagram is constructed based on two tree graphs generated from original polygons. The trees are almost the same as intersection and union trees proposed in Sadahiro (2010) except that the nodes represent polygons.

To construct the trees, we first evaluate the closeness of every pair of polygons by using a function of distances mentioned earlier. One typical option is to use the summation of size and hierarchy distances:

$$\begin{aligned} D\left(P_i, P_j\right) &= D_S\left(P_i, P_j\right) + D_H\left(P_i, P_j\right) \\ &= A\left(P_i\right) - A\left(P_i \cap P_j\right) \\ &= A\left(P_i \setminus P_j\right) \end{aligned} ,$$

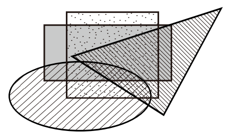
(3)

or its standardized form

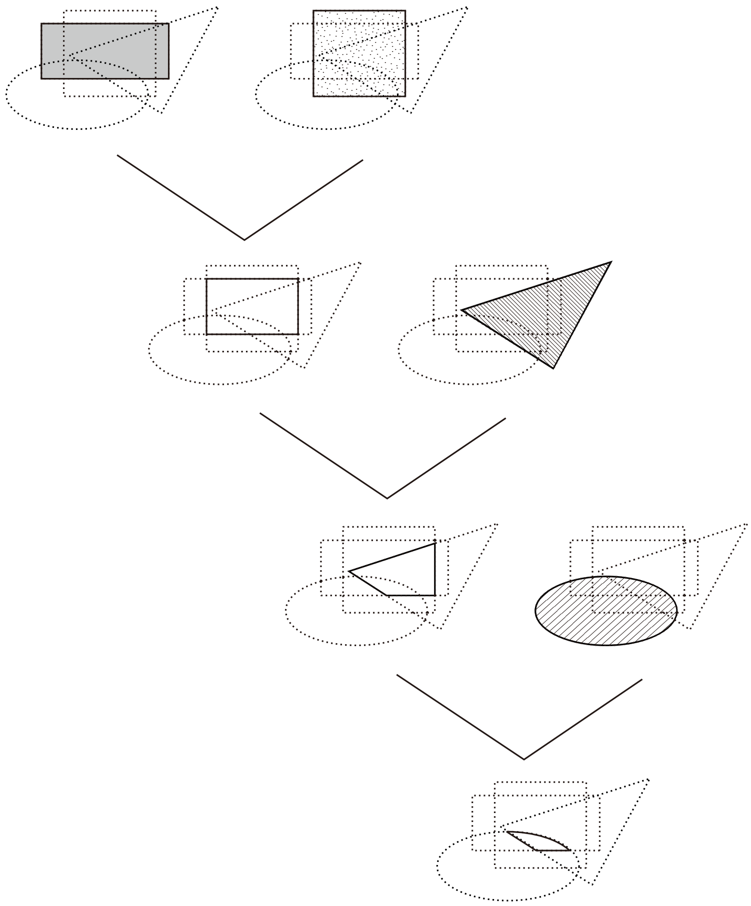
$$
\begin{aligned}
d\left(P_i, P_j\right) &= \frac{D\left(P_i, P_j\right)}{A\left(P_i \cap P_j\right)} \\
&= \frac{A\left(P_i \setminus P_j\right)}{A\left(P_i \cap P_j\right)}
\end{aligned} . \tag{4}
$$

The distance becomes zero only when P_i and P_j are equal under the condition $A(P_i) \geq A(P_j)$. It increases with the hierarchical and size distances between P_i and P_j.

From the given set of polygons Ω, we choose a pair of polygons of the smallest distance and replace them by their intersection (Figure 5). From Ω we again choose two polygons of the smallest distance and replace them by their intersection. This process continues until Ω consists of only one element.



(a)



(b)

Figure 5 Construction of intersection tree from four polygons. (a) Original polygons, (b) tree construction process. The rectangles are chosen first to be replaced by their intersection. The triangle and ellipse are chosen in turn to generate a small polygon at the bottom of the tree. Dotted lines indicate the location of original polygons.

Using union operation instead of intersection operation in the above process yields a union tree. A pair of polygons is replaced by their union. The intersection tree grows downward from original polygons while the union tree extends upward.

Intersection and union trees implicitly assume different hierarchical structure of polygons. Figure 6 show typical examples of hierarchical structures assumed in intersection and union trees and their corresponding tree representations. A big difference between the two structures lies in the property shared by the polygons involved in the trees. In intersection tree, hierarchical structure is based on the place sharing characterized by the smallest polygon at the bottom of the tree. All the polygons share the same property that a sub-region contains the same small polygon. In union tree, on the other hand, all the polygons share the same property that they are contained in the largest polygon at the top of the tree. The former assumes partial overlap of polygons that is not allowed in the latter.

The structure assumed in union tree may be familiar especially to geographers. It leads to the spatial tessellation, a set of regions that are collectively exhaustive and mutually exclusive except for the boundaries. We often find this structure in the real world such as administrative systems and generalized systems of Christaller’s Central Place Theory.

The above difference implies that the trees extract different hierarchical structures from the same set of polygons. Since they detect only the hierarchical structures on which their construction processes assume, whether the generated tree is reliable heavily depends the agreement of the assumption and reality, that is, the hierarchical structure assumed in the tree and that lying in the given set of polygons.

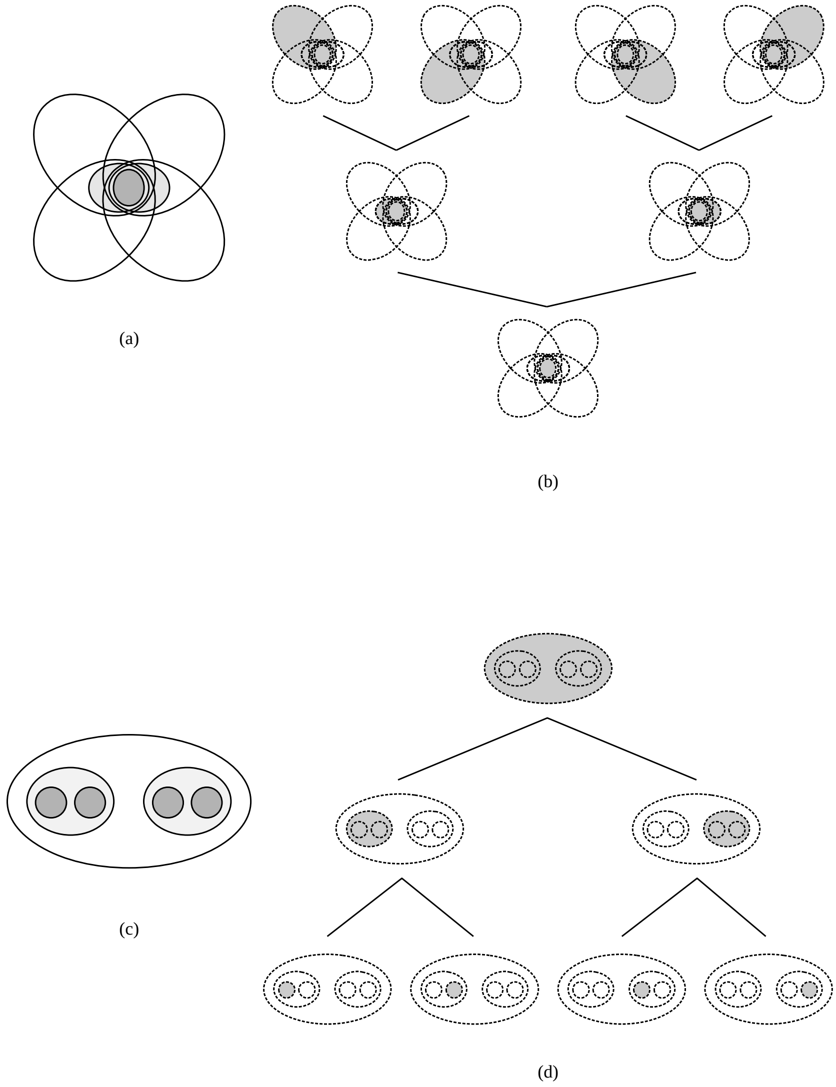


Figure 6 Hierarchical structures and their graph representations implicitly assumed in intersection and union trees. Hierarchical structures of three level polygons assumed in (a)(b) intersection tree, (c)(d) union tree.

3.2.2 Arrangement of polygons

Having obtained intersection and union trees, we combine them into a single graph. Arranging the polygons in size order along the vertical axis, we visualize not only hierarchical but also size relations among polygons.

Figure 7 shows a graph generated from the intersection and union trees of two polygons, where the vertical axis indicates the area of polygons. Similar to Hasse diagram, it shows the hierarchical relation by a chain of links. Polygons connected directly by a single link are completely hierarchical.

As seen in Figure 7, the distances between polygons defined earlier can be easily calculated in the graph. The size distance is the difference in the vertical coordinate between original polygons. The hierarchical distance is the difference in the vertical coordinate between the smaller polygon of original ones and their intersection. It is also given by the difference in the vertical coordinate between the larger polygon of original ones and their union.

If two polygons are not completely but almost hierarchical, the shorter of two links connecting the polygons with their intersection or union becomes negligibly short that the polygons look like connected directly by a single link. This conceals the vagueness and inaccuracy in representation of spatial phenomena that are not critical in exploratory analysis. It is useful especially when we treat natural and cultural phenomena, because we can focus on understanding the global structure of the relations among polygons.

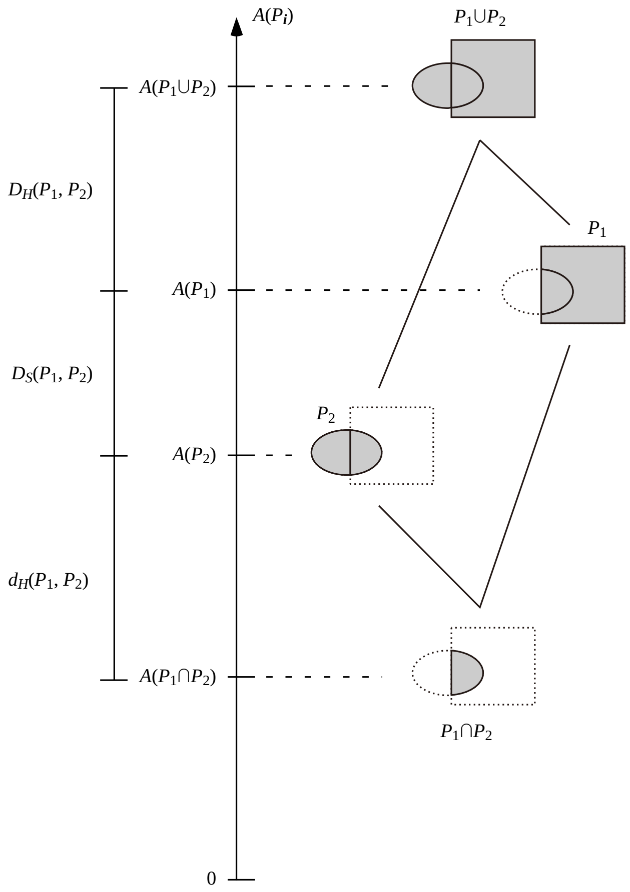


Figure 7 A graph representation generated from two polygons. The vertical axis indicates the area of polygons. The size and hierarchy distances are given by the difference in the vertical coordinate between polygons.

3.2.3 Elimination of redundant links

The graph mentioned above is generated from intersection and union trees that are obtained independently from original polygons. This often yields redundant links, that is, links connecting two nodes that are also reachable by tracing multiple links in the same direction. The transitive property of posets assures that we can eliminate such redundant links without losing any information on the relations among polygons. To obtain a simpler graph representation, we examine every link whether it can be substituted by a set of other links to eliminate all the redundant links. This paper calls the graph obtained after the elimination process a *quasi-Hasse diagram*.

3.3 Typified quasi-Hasse diagram

Quasi-Hasse diagram consists of fewer links and thus simpler than ordinary Hasse diagram. Concerning the total length of links, quasi-Hasse diagram can be further simplified.

Suppose three polygons P_1, P_2, and P_3 as shown in Figure 8a. Though $P_1 \cap P_2 \neq P_3$, this result can occur because the intersection and union trees are constructed independently. Polygons P_1 and P_2 are connected directly with P_3 if P_1 and P_3 are connected during the construction of intersection tree while P_2 and P_3 are connected in the union tree.

This representation, however, is redundant in a sense. The links can be shortened further by inserting intermediate polygon $P_1 \cap P_2$ in the graph and replace the existing links by three new ones as shown in Figure 7b. By this we can minimize the total length of links connecting P_1, P_2, and P_3 with keeping the graph equivalent to the original one.

Similar operation can be applied to the case where two polygons P_2 and P_3 are directly connected to a larger polygon P_1. Inserting $P_2 \cup P_3$ in the graph, we obtain an equivalent graph that consists of shorter links. The process of polygon insertion and link replacement is called *link typification* in this paper.

Given a quasi-Hasse diagram, we calculate the reduction of the length of links obtained by link typification for every triplet of polygons connected directly by two links. We perform link typification on triplets of polygons in the order of the reduction of total link length. Though this does not assure the minimization of the total length of links, it at least yields a simpler representation of the relations among polygons. The original and obtained graphs are called *ordinary* and *typified quasi-Hasse diagrams*, respectively.

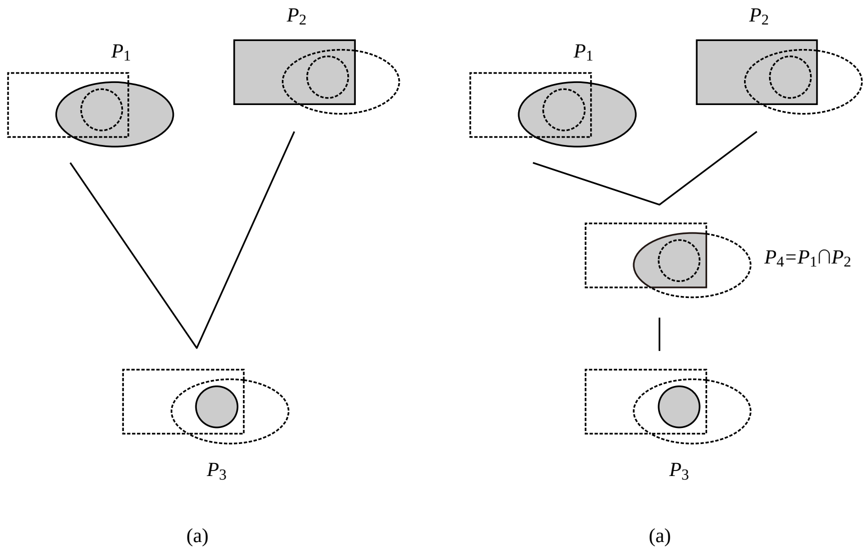


Figure 8 Link typification. (a) Three polygons P_1, P_2, and P_3 connected directly by two links. (b) The total length of links is minimized by inserting $P_1 \cap P_2$ in the graph.

3.4 Simplified quasi-Hasse diagram

In analysis of polygons, we find pairs of polygons that are not completely but almost hierarchical. Such a slight departure from hierarchical relation is often negligible, and those polygons are as important in practice as completely hierarchical ones. In exploratory analysis, it is effective to treat those polygons equivalently as hierarchical ones if the departure is negligibly small.

When polygons are not completely but almost hierarchical, the shorter link connecting the polygons with their intersection or union becomes so short that the polygons look like connected directly by a single link (Figure 9a). We thus replace the indirect links by a single link as shown in Figure 9b. Two polygons are connected directly by a single link, and their intersection or union disappears. This process is called *link simplification* in this paper.

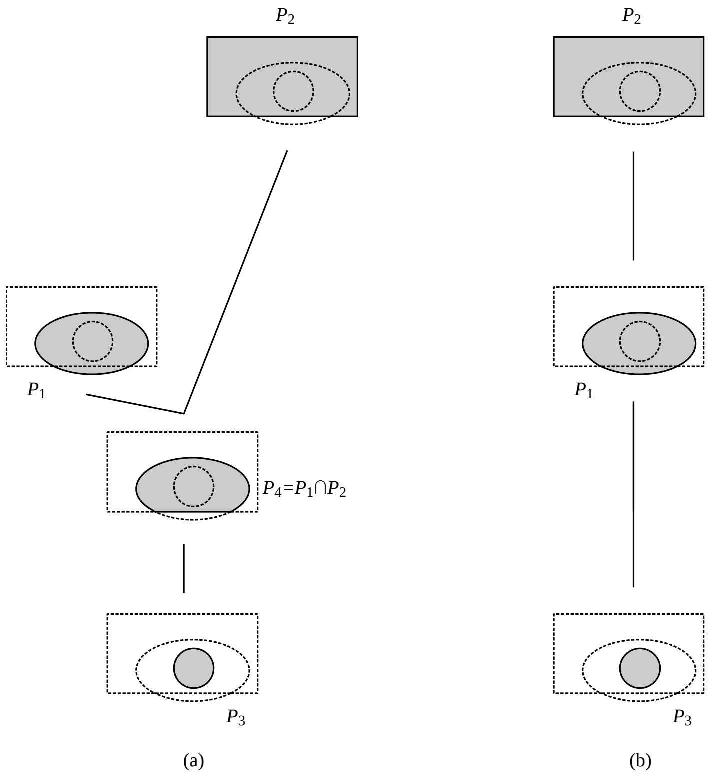


Figure 9 Link simplification. (a) Three polygons P_1, P_2, and P_3 ,where P_1 and P_2 are almost hierarchical. (b) Direct connection of P_1 and P_2 by a single link.

Link simplification permits us to analyze more easily the hierarchical relations among polygons, either complete or almost complete, in exploratory visual analysis. It also helps us understanding the whole structure of the relations among polygons since link simplification yields a simpler graph than the ordinary and typified quasi-Hasse diagrams.

Link simplification is performed as follows. Given two polygons P_i and P_j, we evaluate the length of the link connecting the smaller one and $P_i \cap P_j$ by

$$l_S\left(P_i, P_j\right) = \frac{\min\left(A\left(P_i\right), A\left(P_j\right)\right) - A\left(P_i \cap P_j\right)}{\max\left(A\left(P_i\right), A\left(P_j\right)\right) - A\left(P_i \cap P_j\right)}. \tag{5}$$

We perform link simplification on a pair of polygons in the ascendant order of the length defined above while it is shorter than a predefined value. The obtained graph is called a *simplified quasi-Hasse diagram*.

3.5 Comparison of quasi-Hasse diagrams

Quasi-Hasse diagrams represent the mutual relations among polygons. At local scale, they permit us to detect hierarchical relations among polygons, either single- or multi-level, and to compare the size of polygons. At global scale, quasi-Hasse diagrams allow us to examine the size distribution of polygons and to grasp the whole picture of the hierarchical relations among polygons.

The ordinary, typified and simplified quasi-Hasse diagrams are different in several aspects, and thus the choice depends on the objective of analysis.

The typified quasi-Hasse diagram is simpler than the ordinary one in the sense that the former consists of shorter links than the latter. However, the typified quasi-Hasse diagram has more nodes because link simplification adds new nodes to the graph. Therefore, the typified quasi-Hasse diagram may not be simpler as a whole especially when nodes are visually emphasized than links. The simplified quasi-Hasse diagram, on the other hand, consists of shorter links and fewer nodes so that it is simpler than the ordinary and typified ones. In exploratory spatial analysis, in general, simpler representations are more suitable than complicated ones, especially in detecting spatial patterns. To this end, the simplified quasi-Hasse diagram would be better than the ordinary and typified quasi-Hasse diagrams.

The simplified quasi-Hasse diagram, on the other hand, is an approximated representation of the relations among polygons while the ordinary and typified quasi-Hasse diagrams are correct representations of the relations. Consequently, the simplified quasi-Hasse diagram can be used only if the approximation is permissible. If a slight departure from perfect hierarchy is not negligible, or the strict representation of hierarchical relations is more important than the detection of local spatial patterns, the ordinary or typified quasi-Hasse diagrams should be chosen.

Simplicity in representation increases the flexibility of embedding the graph on the two-dimensional space as a diagram. This permits us to reduce the intersections of links to obtain simpler diagrams. As discussed in lattice drawing (Wille ,1989; Aeschlimann and Schmid, 1992; Freese, 2004; Baixeries et al., 2009), intersection of links makes graphs more complicated and difficult to interpret. On this point, the simplified quasi-Hasse diagram is expected to be more suitable for exploratory spatial analysis.

3.6 Classification of polygons using intersection and union trees

One application of quasi-Hasse diagram is the classification of polygons. Since the diagram is generated intersection and union trees, sub-trees of quasi-Hasse diagram naturally define groups of similar polygons.

Classification can be based on either intersection or union tree. However, the result heavily depends on the tree on which classification is based, because the trees assume different hierarchical structures among polygons. As discussed in Subsection 3.2.1, intersection tree assumes the place sharing among polygons. Consequently, it collects polygons that share the same small polygon (recall Figure 6). Union tree, on the other hand, is based on the division of large areas into small ones. It gathers polygons generated by dividing the same larger one polygon.

Classification is a method of exploratory analysis. Consequently, it inevitably involves interactive and visual analysis. In this sense, one had better to apply link simplification to intersection and union trees before classification. Simple representation improves the efficiency of visual analysis of classification result, say, examination of whether the result is valid in theoretical and practical sense.

4. Application

This section applies the proposed method to analysis of the result of experiment in environmental psychology. The experiment was conducted in November, 2008 in the Department of Urban Engineering at the University of Tokyo. The objective of experiment is to analyze the mental image of Aoyama area in Tokyo, Japan (Figure 10). Aoyama is famous for its shopping streets filled with fashion designers' boutiques, cafes, and upmarket groceries. Museums, art galleries and universities are also located in this area.

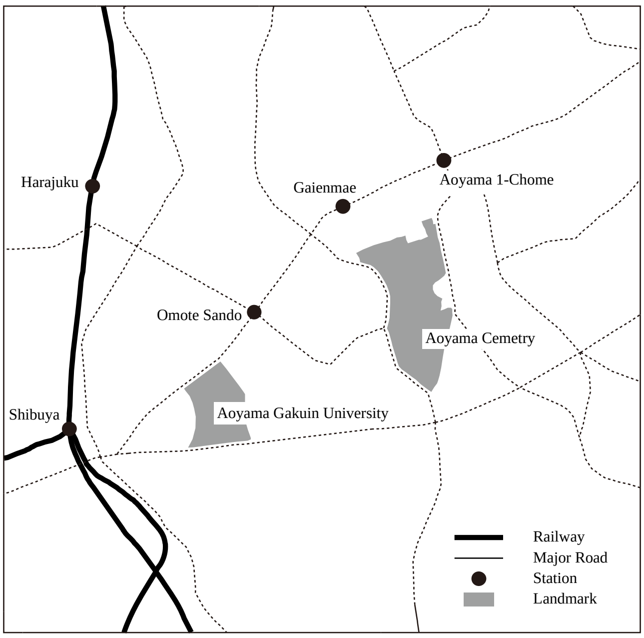


Figure 10 A map of Aoyama area.

In the experiment, 53 undergraduate and 17 graduate students served as subjects. The age ranges from 19 to 22 and 24 to 59 among undergraduate and graduate students, respectively. The subjects were handed a base map of Aoyama area, and asked to draw the extent of Aoyama area in their mind.

As a result, 70 drawings were obtained as shown in Figure 11. As seen in the figure, a wide variety exists in the mental image of Aoyama area. Some students drew small circles around Aoyama Gakuin University, while others drew large circles that cover all the landmarks in this area. Clusters are found around subway stations.

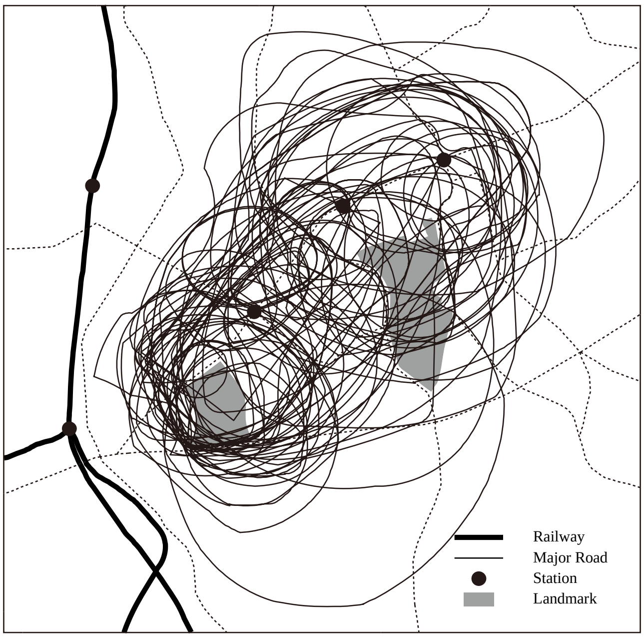


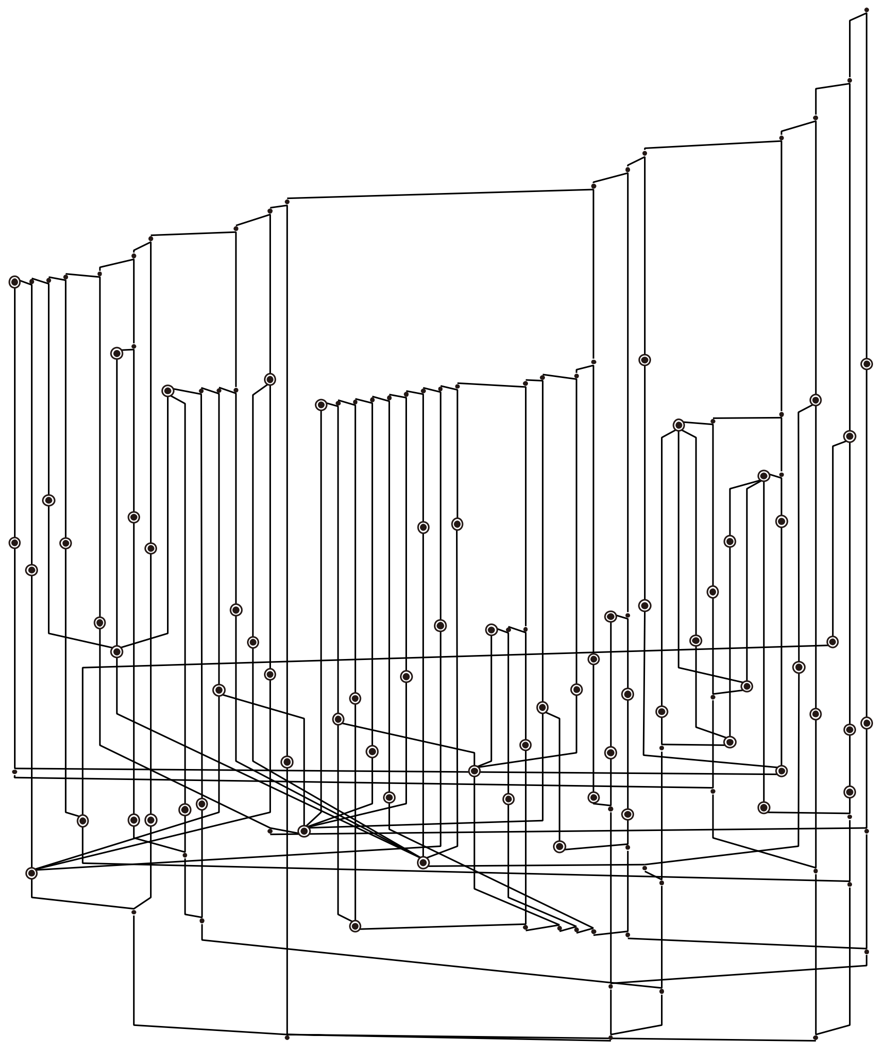
Figure 11 Mental images of Aoyama area drawn by 70 students.

Though Figure 11 provides us a lot of useful information, it is too complicated to analyze visually. We thus applied the method proposed in the previous section to understand the relations among mental images and to classify them into similar groups.

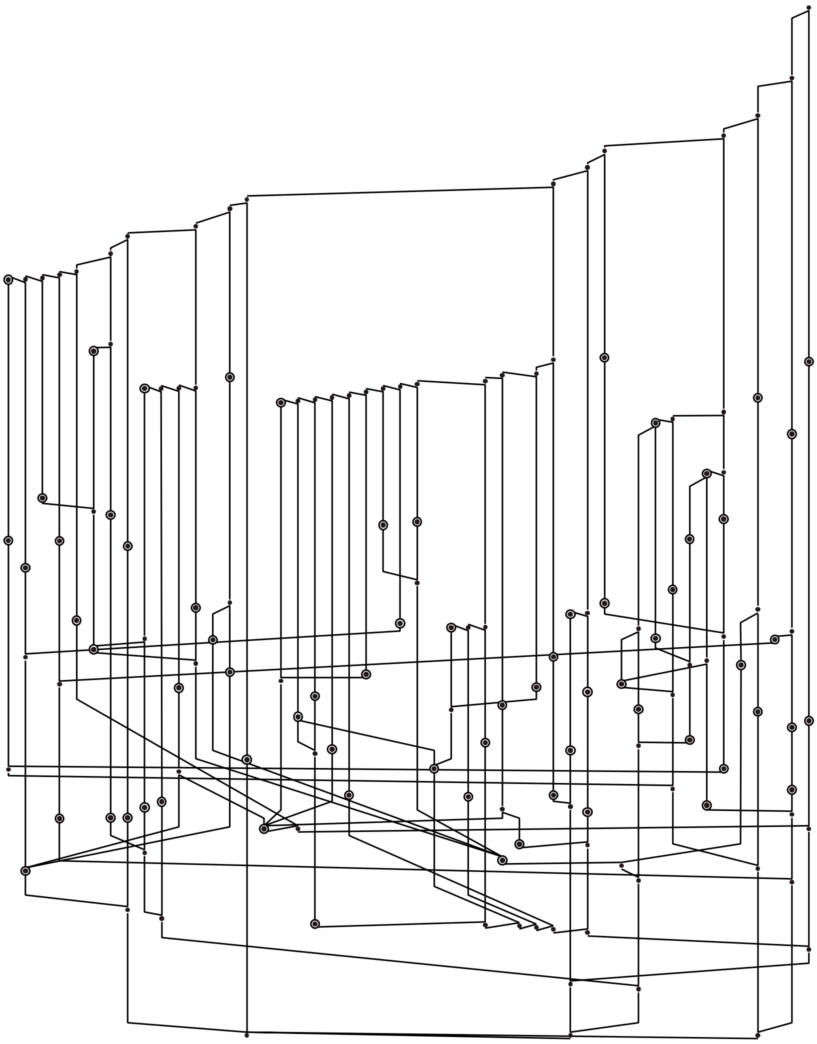
In construction of trees and diagrams, the closeness of polygons has to be evaluated quantitatively. This application adopted the definition of the distance given by Equation (4). To construct simplified Quasi-Hasse diagrams, we performed link simplification until the length of the link given by Equation (6) is shorter than 0.10.

The results are shown in Figure 12. As seen in the figure, the ordinary and typified quasi-Hasse diagrams are similarly complicated while the simplified quasi-Hasse diagram is simpler than the others. This implies that link typification is not effective as link simplification to obtain a simple representation of the relations among mental images. An increase in the number of nodes and links seems to cancel the effect of reduction in link length.

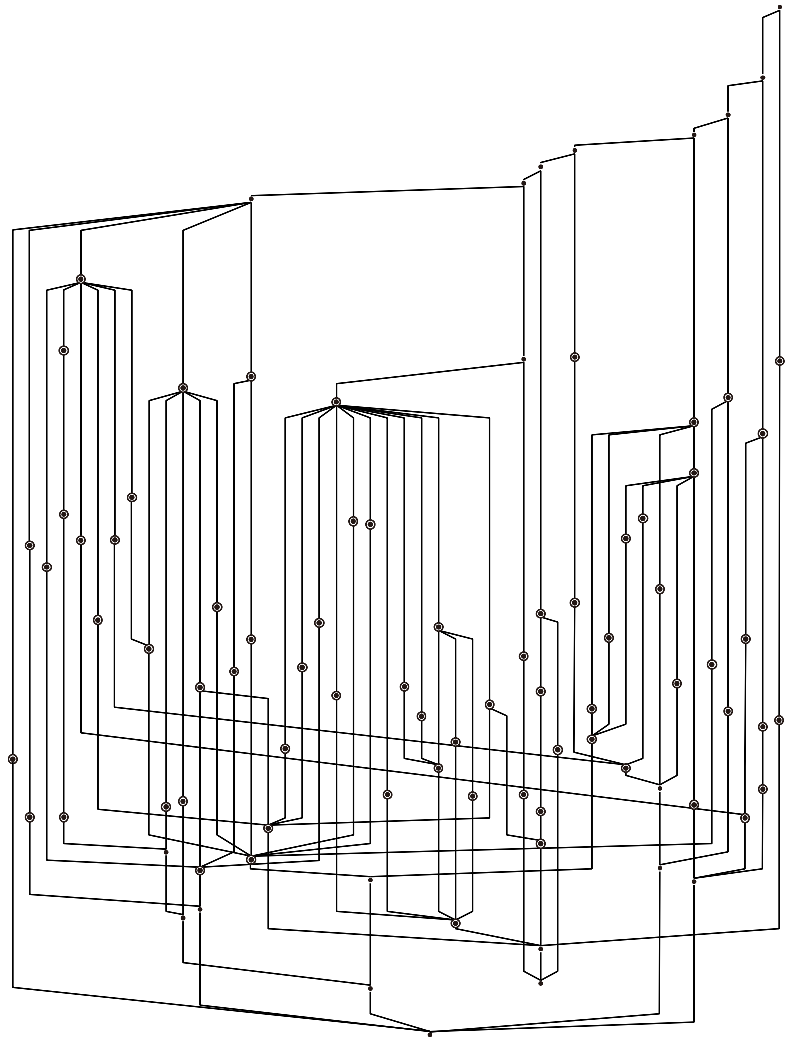
The simplified quasi-Hasse diagram is simpler than the other diagrams, especially in the number of link intersections and stepwise links found in the upper and lower part of diagrams in Figures 12a and 12b. Visual flexibility of simplified quasi-Hasse diagram permits us to reduce link intersections. Stepwise links were replaced by one-rooted links so that it is easier to understand the hierarchical relations among mental images and to find similar groups and their common meet or join.



(a)



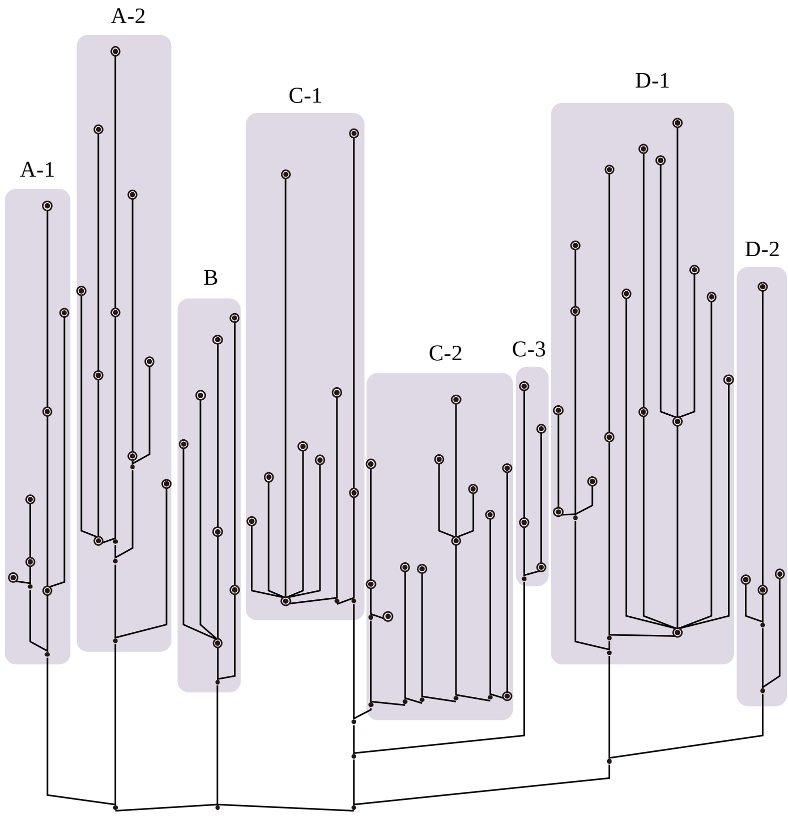
(b)



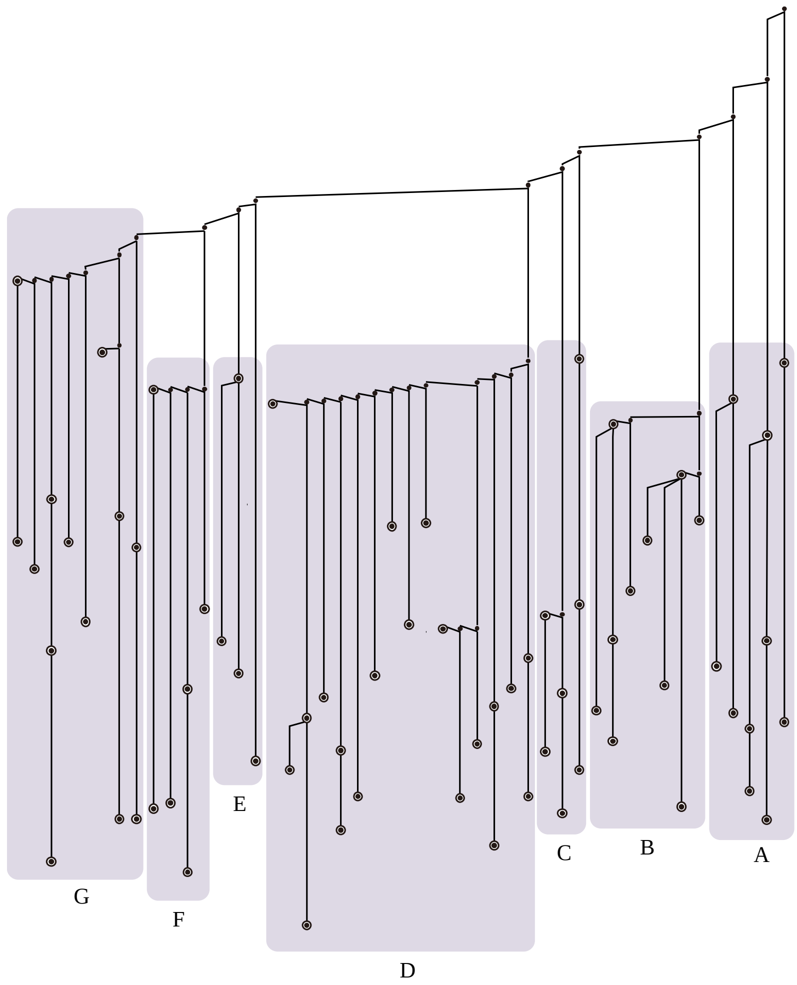
(c)

Figure 12 Relations among mental images of Aoyama area. (a) Ordinary, (b) typified, and (c) simplified quasi-Hasse diagrams. Large circles indicate the original images while small circles are images generated by overlay operations and link typification.

Mental images were classified by using both intersection and union trees (Figure 13). In intersection tree, visual analysis suggests four groups (A, B, C, and D), each of which is further classified into smaller ones to yield eight groups. In union tree, on the other hand, small groups A, B, C, D, E and F are generated successively only from a larger group. This is because the two trees assume different hierarchical structures as mentioned in Section 3.6. Details will be discussed later in this section.



(a)



(b)

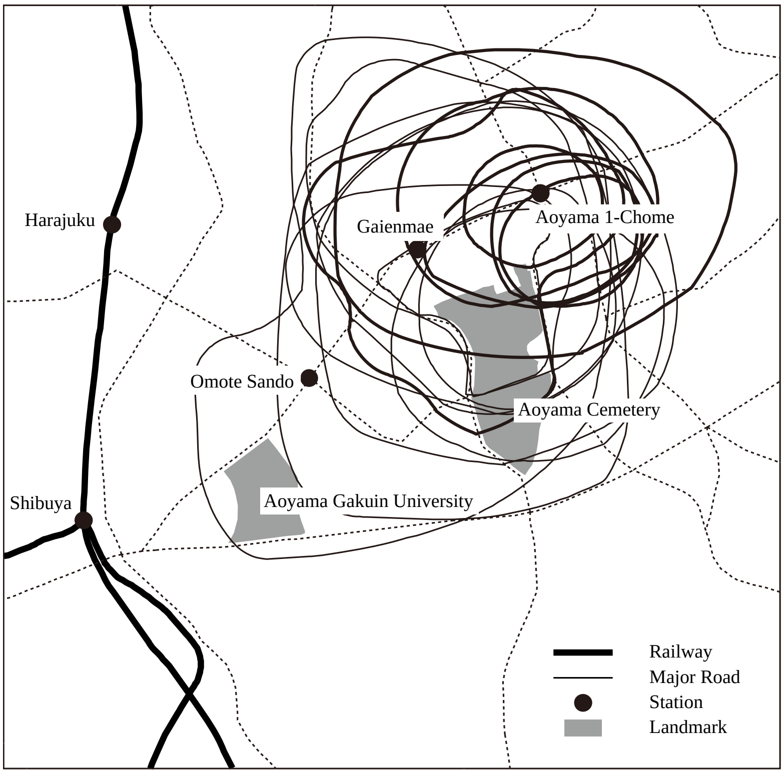
Figure 13 Classification of mental maps by (a) intersection, and (b) union trees. Large circles indicate the original images while small circles are images generated by overlay operations and link typification. Gray shades indicate similar groups of mental images.

Mental maps classified by intersection trees are shown in Figure 14. In each group we find a hierarchical structure such as suggested in Figure 6a. In group A-1 (Figure 14a), for instance, mental images are clustered around Aoyama 1-Chome Station. Group A-2 covers larger areas around Aoyama Cemetery. In group B (Figure 14b), image are densely gathered around Omote Sando Station. Images in group C (Figure 14b) spread around Aoyama Gakuin University. Their intersection near the university corresponds to the smallest circle shown in Figure 6a. At larger scales, we can also find the difference among the images in this group. Group C-1 covers larger areas from the university to the north and east ends of Aoyama area. Images in C-2 are concentrated around the north of the university with covering Omote Sando Station. Images in C-3 are located to the south of those in C-2. Group C is a typical example of multi-level hierarchical structure shown in Figure 6b; all the images in this group share the same landmark (Aoyama Gakuin University) shown at the bottom of the tree, and they are divided into three groups each of which has a common larger area containing the university and other different landmarks such as Omote Sando Station. Groups D-1 and D-2 spread from Omote Sando and Aoyama 1-Chome Stations to the outer boundary of Aoyama area. Images in D-2 are smaller than those in D-1.

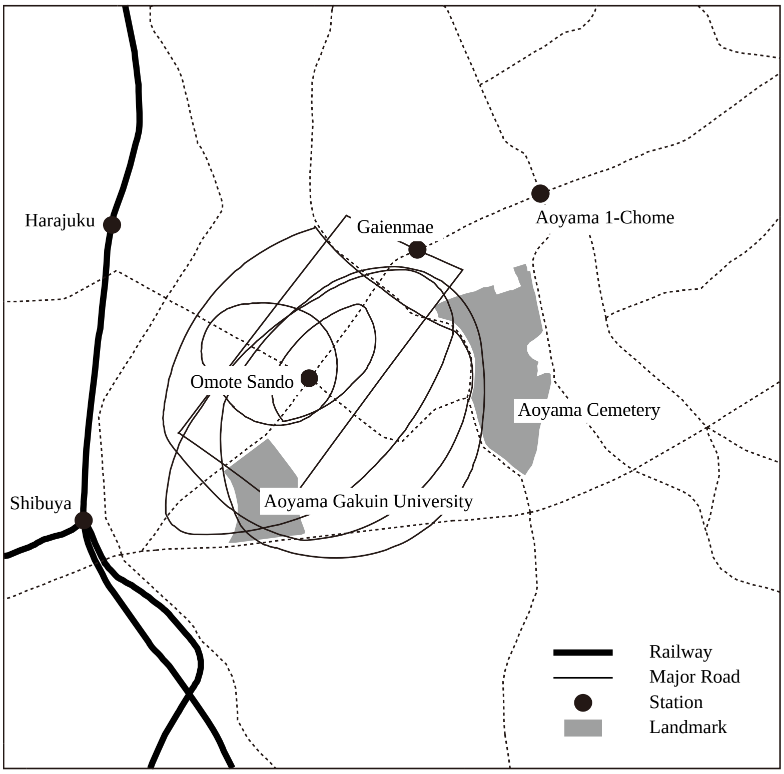
As seen above, mental images spread from certain landmarks in each group as follows:

Group A: Aoyama 1-Chome Station and Aoyama Cemetery
Group B: Omote Sando Station
Group C: Aoyama Gakuin University
Group D: Omote Sando and Aoyama 1-Chome Station

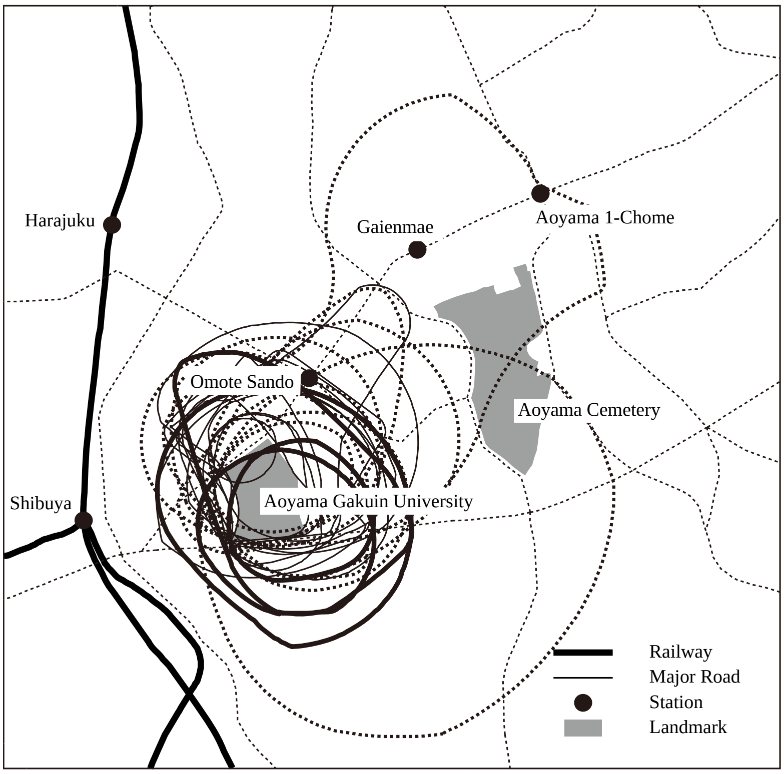
In each group several images are hierarchical as shown in Figure 13a.



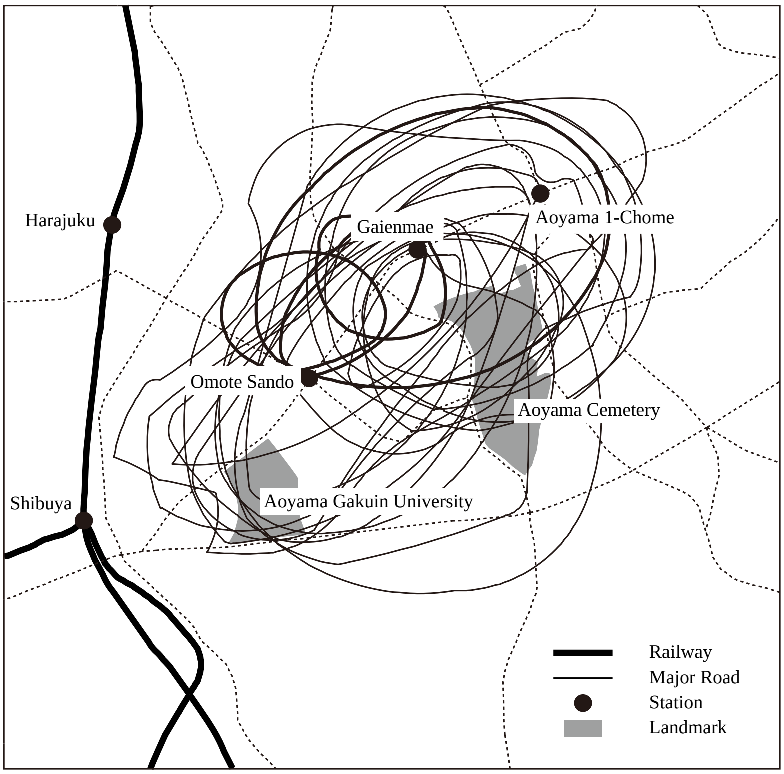
(a)


Harajuku
Gaienmae
Aoyama 1-Chome
Omote Sando
Aoyama Cemetery
Shibuya
Aoyama Gakuin University
Railway
Major Road
Station
Landmark

(b)


Harajuku
Gaienmae
Aoyama 1-Chome
Omote Sando
Aoyama Cemetery
Shibuya
Aoyama Gakuin University
Railway
Major Road
Station
Landmark

(c)


Harajuku
Aoyama 1-Chome
Gaienmae
Omote Sando
Aoyama Cemetery
Shibuya
Aoyama Gakuin University
Railway
Major Road
Station
Landmark

(d)

Figure 14 Mental maps classified by intersection tree. (a) Group A (bold lines: A-1, thin lines: A-2), (b) group B, (c) group C (bold dotted lines: C-1, thin lines: C-2, bold lines: C-3), (d) group D (thin lines: D-1, bold lines: D-2).

The above classification can be partially explained by the attribute of students. Mental maps in group C are drawn by undergraduate students while groups A and D consist of mental maps of graduate students. Young students are familiar with only Aoyama Gakuin University in this area while senior students often visit boutiques, bars and restaurants around Omote Sando and Aoyama 1-Chome Stations.

Frequency of visiting Aoyama area also affects mental image. In general, the mental image of Aoyama area expands with the frequency. Frequent visitors whose maps mostly belong to groups A and D draw large circles. Students who rarely visit Aoyama area drew very small circles around landmarks whose name contains “Aoyama” s uch as Aoyama 1-Chome Station, Aoyama Cemetery and Aoyama Gakuin University.

Let us turn to the union tree shown in Figure 13b. As mentioned earlier, the union tree generates groups A, B, C, D, E and F successively only from a larger group; the whole set is divided into two sets {A} and {B, C, D, E, F, G}, and then the former is divided into {B} and {C, D, E, F, G}, and so forth. Such a tree is generally difficult to interpret and not suitable for classification (Everitt *et al.*, 2009).

A main reason is that the mental images do not have the hierarchical structure assumed in union tree (Figures 6c and 6d). As discussed above, mental images spread from landmarks in Aoyama area. Since variation exists mainly in their size as shown in Figure 6b, union tree did not work successfully in this study.

5. Conclusion

This paper has proposed a new method of analyzing the relations among polygons. The relations between a pair of polygons were described by topological and numerical methods. They were extended to the case of more than two polygons, which is represented as Hasse and quasi-Hasse diagrams. The diagrams permit us to grasp the whole structure of the relations among polygons, to detect spatial patterns in the relations, and to classify the polygons into several groups. The method was applied to analysis of the result of an experiment in environmental psychology. It revealed the properties of each diagram as well as provided useful findings in environmental psychology.

We finally discuss some limitations and extensions of the paper for future research.

First, quantitative evaluation of graph representation needs further discussion. This paper proposed three different types of quasi-Hasse diagrams, which vary in simplicity of representation. In general, simple representations are better than complicated ones for exploratory analysis. On the other hand, approximated representations are incorrect to some extent, which may cause misunderstanding of the relations among polygons. Quantitative evaluation of visual representation should be discussed further in validity, efficiency, and accuracy.

Second, this paper considers only the four relations between a pair of polygons. This is, as mentioned earlier, a first step to analyze the relations of more than two polygons. Wider variation such as those discussed in Egenhofer and Franzosa (1991) and (Cohn et al., 1997) should be incorporated into the proposed method.

Another option of this extension is to evaluate the closeness of polygons in different views. Properties of polygons other than area such as perimeter and shape indices used in shape analysis (Serra, 1982; Stoyan and Stoyan, 1994; Dryden and Mardia, 1998) may be helpful to understand the relations among polygons.

Third, polygons are not necessarily stable over time. Market area of a supermarket changes every day. Mental image changes with experience in the real world. This imposes us two research questions: how do we analyze 1) the relations among polygons of different times, and 2) the change of the relations among different polygons? These problems have to be resolved in future research.

References

Aeschlimann, A. and J. Schmid, 1992. Drawing orders using less ink. Order, 9, 5-13.

Anderson, I. (2002). *Combinatorics of Finite Sets.* Dover Publication.

Anselin, L. (1988). *Spatial Econometrics: Methods and Models.* Kluwer, Dordrecht.

Baixeries, J., Szathmary, L., Valtchev, P., and Godin, R., 2009. Yet a Faster Algorithm for Building the Hasse Diagram of a Concept Lattice. In S. Ferre and S. Rudolph (eds.) *Formal Concept Analysis: Proceedings of the 7th International Conference on Formal Concept Analysis, Lecture Notes in Computer Science*, vol. 5548, pp. 162-177, Springer, New York.

Birkhoff, G., 1979. *Lattice Theory (3rd Ed.)*. American Mathematical Society, Providence, RI.

Christaller, W., 1933. *Die zentralen Orte in Suddeutschland*. Gustav Fischer, Jena.

Cohn, A. G., Bennett, B., Gooday, J., and Gotts, M. M., 1997. Qualitative spatial representation and reasoning with the Region Connection Calculus. *Geoinformatica*, 1, 275-316.

Cohn, A. G., Bennett, B., Gooday, J., and Gotts, N. M., 1997. Representing and reasoning with qualitative spatial relations about regions. In O. Stock (ed.) *Spatial and Temporal Reasoning*, pp. 97-134, Kluwer, Dordrecht, Holland.

Davey, B. A. and Priestley, H. A., 2002. *Introduction to Lattice and Order*, Cambridge: Cambridge University Press.

Egenhofer, M. J., and Franzosa, R. D., 1991. Point set topological spatial relations. *International Journal of Geographical Information Systems*, **5**, 161-174.

Egenhofer, M. J., and Franzosa, R. D., 1994. On the equivalence of topological relations. *International Journal of Geographical Information Systems*, **8**, 133-152.

Egenhofer, M. J., Clementini, E., and Di Felice, P., 1994. Topological relations between regions with holes. *International Journal of Geographical Information Systems*, **8**, 129-144.

Everitt, B. S., Landau S, and Leese, M., 2009. *Cluster Analysis (4th edition)*, Chichester: John Wiley & Sons.

Fotheringham, A. S., Brunsdon, C., and Charlton, M., 2000. *Quantitative Geography: Perspectives on Spatial Data Analysis*, Sage, London.

Fotheringham, A. S., Brunsdon, C., and Charlton, M., 2002. *Geographically Weighted Regression: The Analysis of Spatially Varying Relationships*, Wiley, Chichester.

Freese, R., 2004. Automated Lattice Drawing. In P. Eklund (ed.) *Concept Lattice: Proceedings of the 2nd International Conference on Formal Concept Analysis, Lecture Notes in Artificial Intelligence*, vol. 2961, pp. 112-127, Springer, New York.

Guting, R., 1988. Geo-relational algebra: a model and query language for geometric database systems. In J. Schmidt, S. Ceri and M. Missikoff (eds.) *Advances in Database Technology – EDBT '88, International Conference on Extending Database Technology, Lecture Notes in Computer Science*, vol. 303, pp. 506-527, Springer, New York.

Kainz, W., Egenhofer, M. J., and Greasley, I., 1993. Modeling spatial relations and operations with partially ordered sets. *International Journal of Geographical Information Systems*, **7**, 215-229.

LeSage, J. and Pace, R. K., 2009. *Introduction to Spatial Econometrics*. Chapman & Hall, Boca Raton, FL.

Dryden, I. L. and Mardia, K. V., 1998. *Statistical Shape Analysis*. Wiley, Chichester.

Pemmaraju, S. and Skiena, S., 2003. *Computational Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*. Cambridge University Press, Cambridge.

Randell, D. a., Cui, Z., and Cohn, A. G., 1992. A spatial logic based on regions and connections. In B. Nebel,. W. Swartout and C. rich (eds.), *Proceedings of the 3rd International Conference on Knowledge Representation and Reasoning*, Morgan Kaufmann, Los Altos, CA, pp. 165-176.

Renz, J,. 2002. *Qualitative spatial reasoning with topological information*. Berlin: Springer.

Sadahiro, Y., and Sasaya, T., 2008. Analysis of the relationship among spatial tessellations. *Discussion Paper*, **96**, Department of Urban Engineering, University of Tokyo.

Sadahiro, Y., 2010. Analysis of the spatial relations among point distributions on a discrete space. *International Journal of Geographical Information Science*, to appear.

Serra, J., 1982. *Image Analysis and Mathematical Morphology*. Academic Press, London.

Stock, O. (Ed.), 1997. *Spatial and Temporal Reasoning*, Kluwer, Dordrecht, Holland.

Stoyan, D. and Stoyan, H., 1994. *Fractals, Random Shapes and Point Fields: Methods of Geometrical Statistics*. Wiley, Chichester.

Wille, R., 1989. Lattices in data analysis: how to draw them with a computer. In I. Rival (ed.), *Algorithms and Order*, pp. 33-58, University of Ottawa.

Ward, M., D. and Gleditsch, K. S., 2008. *Spatial Regression Models*, Pion, Thousand Oaks, CA.