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Polygons Generated by Map Overlay Operation: The Case of Convex Polygons

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Abstract

This paper analyzes the number of polygons generated by map overlay. Overlay of maps yields a number of small polygons, and often causes problems in computation, handling, and storage of derived polygons. One method to deal with these problems is to expect the number of derived polygons before executing overlay operation. To this end a stochastic model is proposed by which the expectation of the number of derived polygons is calculated. The major results are summarized as follows: 1) the number of polygons generated by map overlay depends on the number of original polygons and their perimeter; 2) the number of derived polygons increases with the perimeter of original polygons; 3) McAlpine and Cook (1971)'s conjecture on the number of derived polygons is underestimate; 4) the number of polygons generated by overlay of maps having the same lattice system is proportional to the square of the number of overlaid maps and the number of polygons on the map, and inversely proportional to the area-perimeter ratio of the unit cell.

1 Introduction

Map overlay is one of the fundamental operations of GIS. Overlay of vector maps creates a new map on which generated polygons inherit attributes from the original maps, and enables us to perform spatial analysis on the integrated data. Consider two maps, for instance, a census map reporting socioeconomic data and a landcover map based on a square lattice. Each polygon generated by overlay of these maps has at least two attributes, say, population count and landcover type. Therefore, the new map may reveal an interesting relationship between these variables.

In general, map overlay increases the number of polygons. This is not problematic if overlaid maps have only a few polygons. However, overlay of maps containing millions of polygons may yield tens of millions of polygons, where computation, handling and storage cost cannot be ignored. Computation of polygon intersection takes time and derived polygons occupy much space in storage devices. It could even happen that a new map cannot be saved on disks.

One method to deal with these problems is to understand what determines the number of derived polygons and to expect it before executing overlay operation. If the number of derived polygons is expected in advance, the problem can be solved by partitioning maps before overlay or adding storage devices. McAlpine and Cook (1971) were among the first to examine the number of polygons generated by map overlay. They randomly thrown a single hexagon over a hexagonal lattice, counted the number of derived polygons, and proposed an approximate expression of the average number of derived polygons. Their experiments dealt with map overlay where no spatial correlation exists between maps, though they intended a spatially-correlated case - the spurious polygon problem that occurs when two differently digitized maps are overlaid (MacDougall 1975; Chrisman 1987; Goodchild and Gopal 1989; Burrough and McDonnell 1998). Goodchild (1978), on the other hand, discussed the spurious polygon problem more directly. He proposed a model of digitizing process of smoothly curved lines, and calculated the expectation of the number of spurious polygons generated by overlay of differently digitized maps. The model provides not only the expectation but also the minimum and maximum of the number of derived polygons.

This paper is on the line of McAlpine and Cook; we consider the number of derived polygons when spatially-independent maps, say, zipcode boundaries and a square lattice, are overlaid. Unfortunately, the equation proposed by McAlpine and Cook underestimates the expectation of the number of polygons since it is based on numerical experiments rather than theoretical consideration. We show the correct expectation in the following two sections. In addition, McAlpine and Cook's analysis is limited to the case of hexagonal lattices. We extend the discussion to a more general case where original polygons are convex, not necessarily hexagonal.

2 Number of Polygons Generated by Overlay of Two Maps

Let us suppose two map layers L_1 and L_2. Each layer contains a region of the same shape which can cover a plane by its lattice, say, a triangle, a square, or a parallelogram. The region on the layer L_i is denoted by R_i. The region R_i consists of m_i convex polygons Z_{i1}, Z_{i2}, ..., Z_{im_i}, say, Voronoi polygons (Okabe *et al*. 1992) or rectangles (Figure 1). The area and perimeter of Z_{ij} are denoted by A_{ij} and P_{ij}, respectively.

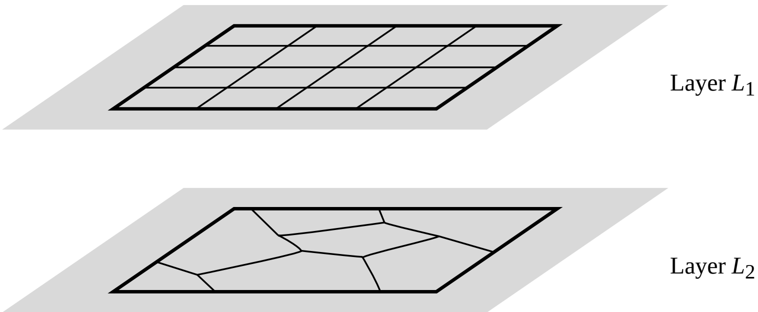

Figure 1 An example of layers L_1 and L_2.

We assume that R_i is surrounded by its copies that have the same configuration of polygons (Figure 2). This assumption is called *periodic continuation* (Ripley 1981; Stoyan and Stoyan 1995; Sadahiro 1999).

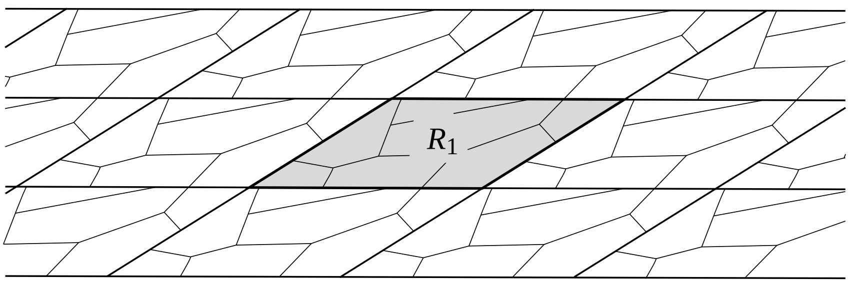

Figure 2 The original region R_1 and its surrounding copies.

The number of polygons generated by map overlay depends on not only the geometry of original polygons but also the relative location and orientation of layers. In order to average the effects of the latter, we propose a stochastic model of map overlay in which the layer L_2 is dropped randomly on L_1 in such a way that the region R_2 intersects R_1 (Figure 3). In other words, the model considers all possible relative locations and orientations of the two maps.

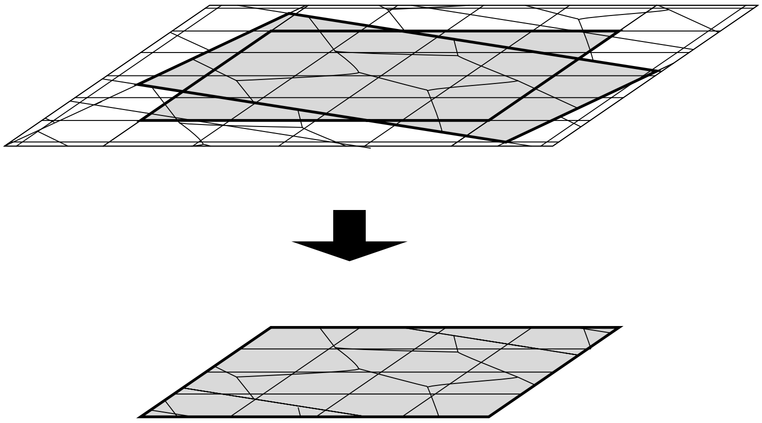

Figure 3 Overlay of the layer L_2 on L_1 and derived polygons.

Using this model, we calculate the expectation of the number of derived polygons in R_1 as follows. Let n_{1i} be the number of derived polygons in Z_{1i}. The number of derived polygons in R_1 is given by

$$N = \sum_i n_{1i} . \tag{1}$$

The probability that the polygon Z_{2j} is overlaid on Z_{1i} is

$$\Pr\left(Z_{1i} \cap Z_{2j} \neq \varnothing\right) = \frac{2\pi\left(A_{1i} + A_{2j}\right) + P_{1i}P_{2j}}{2\pi A} \tag{2}$$

(see Appendix 1 for details), where

$$A = \sum_i A_{1i} = \sum_j A_{2j} . \tag{3}$$

The expectation of n_{1i} is written as

$$\begin{aligned} \mathrm{E}\left[n_{1i}\right] &= \sum_j \Pr\left(Z_{1i} \cap Z_{2j} \neq \varnothing\right) \\ &= \frac{1}{2\pi A}\left(2\pi m_2 A_{1i} + 2\pi A + P_{1i}\sum_j P_{2j}\right) \end{aligned} . \tag{4}$$

From equations 1 and 4 we have

$$\begin{aligned} \mathrm{E}[N] &= \sum_i \mathrm{E}\left[n_{1i}\right] \\ &= m_1 + m_2 + \frac{1}{2\pi A}\sum_i P_{1i}\sum_i P_{2i} \end{aligned} . \tag{5}$$

Equation 5 indicates that the number of derived polygons depends on not only the number of original polygons but also their perimeter. It also shows that overlay of polygons having complicated or elongated shapes creates more polygons than that of rounded polygons. This result seems consistent with intuition.

We then discuss equation 5 in more specific cases where both sets of polygons constitute lattices, that is, overlay of two lattices. This would make the effects of the number and shape of original polygons more clear and easy to understand.

Suppose that the two regions R_1 and R_2 consist of lattices. Let A_i and P_i be the area and perimeter of the unit cell of R_i, respectively. Then equation 5 becomes

$$\mathrm{E}[N] = m_1 + m_2 + 2\sqrt{\frac{m_1 m_2}{\gamma_1 \gamma_2}}, \tag{6}$$

where γ_i is the area-perimeter ratio of the unit cell of R_i (Stoyan and Stoyan 1994) which is defined by

$$\gamma_i = \frac{4\pi A_i}{P_i^2} \tag{7}$$

($\gamma_i \leq 1$). Equation 6 again shows that overlay of polygons having longer perimeter yields more polygons.

Using equation 6 we can calculate the expectation of the number of derived polygons when regular hexagonal lattices are overlaid, the case examined numerically by McAlpine and Cook. When the lattices are regular hexagonal, then

$$\gamma_i = \frac{\sqrt{3}}{6}\pi \ (i=1,\ 2). \tag{8}$$

Substituting equation 8 into equation 6 we have

$$\begin{aligned} \mathrm{E}[N] &= m_1 + m_2 + \frac{4\sqrt{3}}{\pi}\sqrt{m_1 m_2} \\ &\approx m_1 + m_2 + 2.205\sqrt{m_1 m_2} \end{aligned}. \tag{9}$$

Comparing this equation with McAlpine and Cook's conjecture

$$\mathrm{E}[N] = m_1 + m_2 + 2\sqrt{m_1 m_2}, \tag{10}$$

we notice that equation 10 underestimates the number of derived polygons.

Overlay of rectangular lattices creates more polygons. Let us suppose that both regions R_1 and R_2 consist of rectangular lattices. The v/h ratio of the unit cell of R_i is denoted by c_i ($c_i \geq 1$). The area-perimeter ratio is then represented as

$$\gamma_i = \frac{\pi c_i}{(c_i + 1)^2} \tag{11}$$

Substitution of equation 11 into equation 6 yields

$$\mathrm{E}[N] = m_1 + m_2 + \frac{2(c_1 + 1)(c_2 + 1)}{\pi\sqrt{c_1 c_2}}\sqrt{m_1 m_2}. \tag{12}$$

The expectation E[N] is an increasing function of c_i, and it is minimized when c_1=c_2=1. This implies that the number of derived polygons increases monotonously as the unit cell of lattices becomes elongated. Figure 4 shows the relationship between E[N] and the v/h ratio c_i of rectangles.

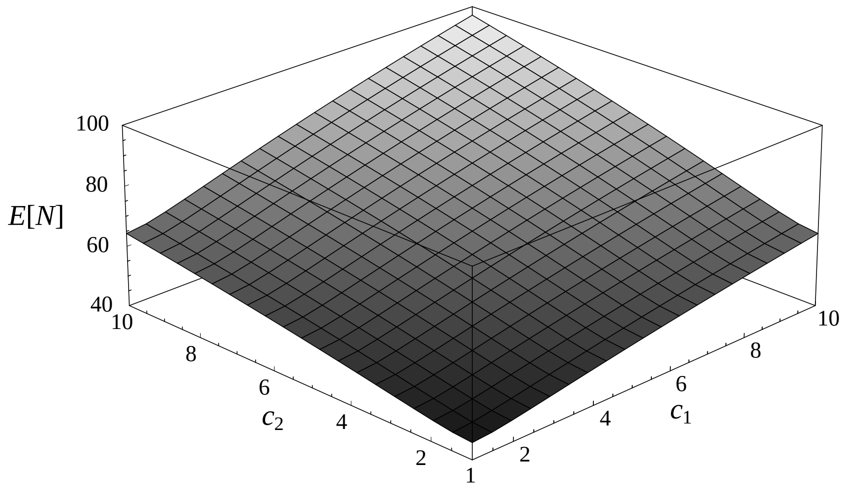

Figure 4 The relationship between E[N] and the v/h ratio c_i of rectangles (m_1=m_2=10).

3 Number of Polygons Generated by Overlay of More Than Two Maps

In the previous section we proposed a method for evaluating the expectation of the number of polygons generated by overlaying two maps. We next extend the methodology to the case of overlaying more than two maps.

Suppose K map layers L_1, L_2, ..., L_K each of which contains a region of the same shape denoted by R_1, R_2, ..., R_K respectively. The region R_i consists of m_i convex polygons Z_{i1}, Z_{i2}, ..., Z_{im_i}, and the area and perimeter of Z_{ij} is denoted by A_{ij} and P_{ij}, respectively. Each region is surrounded by its copies having the same configuration of polygons (periodic assumption).

The layers L_2, L_3, ..., L_K are dropped randomly in such a way that all the regions R_2, R_3, ..., R_K intersect R_1. We denote the number of derived polygons in R_1 by N, and calculate its expectation.

Let us consider the probability that the polygons Z_{2i_2} Z_{3i_3}, ..., Z_{Ki_K} are overlaid on the polygon Z_{1i_1}. This probability is given by

$$\Pr\left(Z_{1i_1} \cap Z_{2i_2} \cap \cdots \cap Z_{Ki_K} \neq \varnothing\right) = \frac{2\pi \sum_j \prod_{k \neq j} A_{ki_k} + \sum_j \sum_{k \neq j} P_{ji_j} P_{ki_k} \prod_{l \neq j,k} A_{li_l}}{2\pi A^{K-1}} \tag{13}$$

(see Appendices 2 and 3 for details), where

$$A = \sum_i A_{1i} = \cdots = \sum_i A_{Ki} . \tag{14}$$

The expectation of N is given by

$$\mathrm{E}[N] = \sum_{i_1,\ldots,i_K} \Pr\left(Z_{1i_1} \cap Z_{2i_2} \cap \cdots \cap Z_{Ki_K} \neq \varnothing\right). \tag{15}$$

Substituting equation 13 into equation 15 we have

$$\mathrm{E}[N] = \sum_i m_i + \frac{1}{2\pi A}\sum_i \sum_{j\neq i}\left(\sum_k P_{ik}\sum_k P_{jk}\right). \tag{16}$$

Similar to equation 5, equation 16 shows that the number of derived polygons depends on the number of original polygons and their perimeter.

We then proceed to more specific cases where all the regions R_1, R_2, ..., R_K consist of lattices. Let γ_i be the area-perimeter ratio of the unit cell of R_i. Equation 16 becomes

$$\mathrm{E}[N] = \sum_i m_i + 2\sum_i \sum_{j\neq i}\sqrt{\frac{m_i m_j}{\gamma_i \gamma_j}}\,. \tag{17}$$

For regular hexagonal lattices, substitution of equation 8 into equation 17 yields

$$\begin{aligned}\mathrm{E}[N] &= \sum_i m_i + \frac{4\sqrt{3}}{\pi}\sum_i \sum_{j\neq i}\sqrt{m_i m_j} \\ &\approx \sum_i m_i + 2.205\sum_i \sum_{j\neq i}\sqrt{m_i m_j}\end{aligned}. \tag{18}$$

Again McAlpine and Cook's conjecture

$$\begin{aligned}\mathrm{E}[N] &= \left(\sum_i \sqrt{m_i}\right)^2 \\ &= \sum_i m_i + 2\sum_i \sum_{j\neq i}\sqrt{m_i m_j}\end{aligned} \tag{19}$$

is found to be underestimate.

If all the layers have the same lattice system, equation 17 becomes

$$\mathrm{E}[N] = Km\left\{1 + \frac{K-1}{\gamma}\right\}, \tag{20}$$

where m is the number of polygons in R_i and γ is the area-perimeter ratio of the unit cell. Equation 20 shows that the number of derived polygons is proportional to m, the number of polygons on the layer. Furthermore, if the number of layers is sufficiently large, equation 20 can be approximated as

$$\mathrm{E}[N] \approx \frac{K^2 m}{\gamma}. \tag{21}$$

This equation indicates that the number of derived polygons is proportional to the square of the number of overlaid layers, and inversely proportional to the area-perimeter ratio of the unit cell.

4. Conclusions

We have analyzed the number of polygons generated by map overlay, and obtained its expectation using a stochastic model. Major results are summarized as follows:

1) The number of polygons generated by map overlay depends on the number of original polygons and their perimeter (equation 16).

2) The number of derived polygons increases with the perimeter of original polygons (equation 16). Therefore, overlay of maps containing polygons of elongated shapes may yield numerous polygons.

3) McAlpine and Cook's conjecture on the expectation of the number of derived polygons is underestimate even for the overlay of regular hexagonal lattices. The correct expectation is given by equations 9 and 18.

4) The number of polygons generated by overlay of maps having the same lattice system is proportional to the square of the number of overlaid maps and the number of polygons on the map, and inversely proportional to the area-perimeter ratio of the unit cell. (equation 21).

In this paper we have dealt with overlay of convex polygons where no spatial correlation exists between maps. Convex polygons have rather simple boundaries, thus overlay operation usually yields fewer polygons than that of nonconvex polygons. Moreover, if overlaid polygons are spatially correlated, it would create numerous spurious polygons (Goodchild 1978; Veregin 1989; Burrough and McDonnell 1998). Therefore, the results obtained in this paper such as equations 5 and 16 can be also regarded as showing a lower boundary of the number of polygons generated by overlay of nonconvex or spatially-correlated polygons.

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Appendix 1 Overlay of Two Maps

This section describes the probability that a polygon on a layer intersects a polygon on another layer. Let Z_1 and Z_2 be convex polygons on layers L_1 and L_2, respectively. The area and perimeter of Z_i are denoted by A_i and P_i, respectively. The measure of all convex sets congruent to Z_2 that intersect Z_1 is given by

$$\begin{aligned} m(Z_2; Z_1 \cap Z_2 \neq \varnothing) &= \int_{Z_1 \cap Z_2 \neq \varnothing} dZ_2 \\ &= 2\pi(A_1 + A_2) + P_1 P_2 \end{aligned} \tag{A 1}$$

(Santaló, 1976). The measure of all sets of regions congruent to R_2 intersecting R_1 is written as

$$m(R_2; R_1 \cap R_2 \neq \varnothing) = 2\pi A. \tag{A 2}$$

Therefore, we have

$$\begin{aligned} \Pr(Z_1 \cap Z_2 \neq \varnothing) &= \frac{m(Z_2; Z_1 \cap Z_2 \neq \varnothing)}{m(R_2; R_1 \cap R_2 \neq \varnothing)} \\ &= \frac{2\pi(A_1 + A_2) + P_1 P_2}{2\pi A} \end{aligned}. \tag{A 3}$$

Appendix 2 Overlay of Three Maps

Let Z_1, Z_2, and Z_3 be convex polygons on layers L_1, L_2, and L_3 respectively. The area and perimeter of Z_i are denoted by A_i and P_i, respectively. The measure of all pairs of polygons congruent to Z_2 and Z_3 that intersect Z_1 is given by

$$\begin{aligned} m(Z_2, Z_3; Z_1 \cap Z_2 \cap Z_3 \neq \varnothing) &= \int_{Z_1 \cap Z_2 \cap Z_3 \neq \varnothing} dZ_2 \wedge dZ_3 \\ &= \int_{Z_1 \cap Z_2 \neq \varnothing} \int_{Z_1 \cap Z_2 \cap Z_3 \neq \varnothing} dZ_3 \, dZ_2 \end{aligned}. \tag{A 4}$$

Let us first fix Z_1 and Z_2, and denote the area and perimeter of $Z_1 \cap Z_2$ as a_{12} and p_{12}, respectively. Then we have

$$\int_{Z_1 \cap Z_2 \cap Z_3 \neq \varnothing} dZ_3 = 2\pi(a_{12} + A_3) + p_{12} P_3. \tag{A 5}$$

Substitution of equation A 5 into equation A 4 yields

$$\begin{aligned} m(Z_2, Z_3; Z_1 \cap Z_2 \cap Z_3 \neq \varnothing) &= \int_{Z_1 \cap Z_2 \neq \varnothing} \left\{2\pi(a_{12} + A_3) + p_{12} P_3\right\} dZ_2 \\ &= 2\pi \left\{ \int_{Z_1 \cap Z_2 \neq \varnothing} a_{12} dZ_2 + A_3 \int_{Z_1 \cap Z_2 \neq \varnothing} dZ_2 \right\} + P_3 \int_{Z_1 \cap Z_2 \neq \varnothing} p_{12} dZ_2 \end{aligned}. \tag{A 6}$$

Using

$$\int_{Z_1 \cap Z_2 \neq \varnothing} a_{12} dZ_2 = 2\pi A_1 A_2, \tag{A 7}$$

$$\int_{Z_1 \cap Z_2 \neq \varnothing} p_{12} dZ_2 = 2\pi\left(A_1 P_2 + A_2 P_1\right) \tag{A 8}$$

(Santaló, 1976), and equation A 1, we have

$$\begin{aligned} m\left(Z_2, Z_3; Z_1 \cap Z_2 \cap Z_3 \neq \varnothing\right) &= 2\pi\left[2\pi A_1 A_2 + A_3\left\{2\pi\left(A_1 + A_2\right) + P_1 P_2\right\}\right] + 2\pi P_3\left(A_1 P_2 + A_2 P_1\right) \\ &= (2\pi)^2\left\{A_1 A_2 + A_2 A_3 + A_3 A_1\right\} + 2\pi\left(P_1 P_2 A_3 + A_1 P_2 P_3 + P_1 A_2 P_3\right) \end{aligned} . \tag{A 9}$$

The measure of all pairs of regions congruent to R_2 and R_3 intersecting R_1 is written as

$$m\left(R_2, R_3; R_1 \cap R_2 \neq \varnothing, R_1 \cap R_3 \neq \varnothing\right) = (2\pi A)^2 . \tag{A 10}$$

Using equations A 9 and A 10 we obtain

$$\begin{aligned} \Pr\left(Z_1 \cap Z_2 \cap Z_3 \neq \varnothing\right) &= \frac{m\left(Z_2; Z_1 \cap Z_2 \cap Z_3 \neq \varnothing\right)}{m\left(R_2, R_3; R_1 \cap R_2 \neq \varnothing, R_1 \cap R_3 \neq \varnothing\right)} \\ &= \frac{2\pi\left(A_1 A_2 + A_2 A_3 + A_3 A_1\right) + P_1 P_2 A_3 + A_1 P_2 P_3 + P_1 A_2 P_3}{2\pi A^2} \end{aligned} . \tag{A 11}$$

Appendix 3 Overlay of More Than Three Maps

Let us consider an overlay of K maps. Let Z_1, Z_2, ..., Z_K be convex polygons on layers L_1, L_2, ..., L_K respectively. The area and perimeter of Z_i are denoted by A_i and P_i, respectively. The measure of all sets of polygons congruent to Z_2, Z_3, ..., Z_K that intersect Z_1 is given by

$$m\left(Z_2, Z_3 \cdots, Z_K; Z_1 \cap Z_2 \cap \cdots \cap Z_K \neq \varnothing\right) = \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_K \neq \varnothing} dZ_2 \wedge dZ_3 \cdots \wedge dZ_K . \tag{A 12}$$

For convenience, we rewrite the above measure as m_K, that is,

$$m_K = \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_K \neq \varnothing} dZ_2 \wedge dZ_3 \cdots \wedge dZ_K . \tag{A 13}$$

We first fix Z_1, Z_2, ..., $Z_{K\text{-}1}$ in the above equation, and denote the area and perimeter of $Z_1 \cap Z_2 \cap Z_3 \cap$ ... $Z_{K\text{-}1}$ as $a_{123...K\text{-}1}$ and $p_{123...K\text{-}1}$ respectively. Then equation A 12 becomes

$$\begin{aligned} m_K &= \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_{K-1} \neq \varnothing} \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_K \neq \varnothing} dZ_K \quad dZ_2 \wedge dZ_3 \cdots \wedge dZ_{K-1} \\ &= \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_{K-1} \neq \varnothing} \left\{2\pi\left(a_{123 \cdots K-1} + A_K\right) + p_{123 \cdots K-1} P_K\right\} dZ_2 \wedge dZ_3 \cdots \wedge dZ_{K-1} \end{aligned} . \tag{A 14}$$

For convenience we define two variables as follows:

$$a_K = \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_K \neq \varnothing} a_{123 \cdots K} dZ_2 \wedge dZ_3 \cdots \wedge dZ_K , \tag{A 15}$$

$$p_K = \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_K \neq \varnothing} p_{123 \cdots K} dZ_2 \wedge dZ_3 \cdots \wedge dZ_K . \tag{A 16}$$

Substituting equations A 15 and A 16 into equation A 14 we obtain

$$
\begin{aligned}
m_K &= \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_{K-1} \neq \emptyset} \left\{ 2\pi\left(a_{123\cdots K-1} + A_K\right) + p_{123\cdots K-1} P_K \right\} dZ_2 \wedge dZ_3 \cdots \wedge dZ_{K-1} \\
&= 2\pi \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_{K-1} \neq \emptyset} a_{123\cdots K-1} dZ_2 \wedge dZ_3 \cdots \wedge dZ_{K-1} \\
&\quad + 2\pi A_K \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_{K-1} \neq \emptyset} dZ_2 \wedge dZ_3 \cdots \wedge dZ_{K-1} \\
&\quad + P_K \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_{K-1} \neq \emptyset} p_{123\cdots K-1} dZ_2 \wedge dZ_3 \cdots \wedge dZ_{K-1} \\
&= 2\pi a_{K-1} + P_K p_{K-1} + 2\pi A_K m_{K-1}
\end{aligned}
. \qquad \text{(A 17)}
$$

Fixing Z_1, Z_2, ..., $Z_{K\text{-}2}$ in equation A 15 we have

$$
\begin{aligned}
a_K &= \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_K \neq \emptyset} a_{123\cdots K} dZ_2 \wedge dZ_3 \cdots \wedge dZ_K \\
&= \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_{K-1} \neq \emptyset} \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_K \neq \emptyset} a_{123\cdots K} dZ_K dZ_2 \wedge dZ_3 \cdots \wedge dZ_{K-1} \\
&= 2\pi A_K \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_{K-1} \neq \emptyset} a_{123\cdots K-1} dZ_2 \wedge dZ_3 \cdots \wedge dZ_{K-1} \\
&= 2\pi A_K a_{K-1}
\end{aligned}
. \qquad \text{(A 18)}
$$

Thus

$$
\begin{aligned}
a_K &= (2\pi)^{K-1} A_K A_{K-1} \cdots A_1 \\
&= (2\pi)^{K-1} \prod_{i=1}^{K} A_i
\end{aligned}
. \qquad \text{(A 19)}
$$

Equation 16 becomes

$$
\begin{aligned}
p_K &= \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_K \neq \emptyset} p_{123\cdots K} dZ_2 \wedge dZ_3 \cdots \wedge dZ_K \\
&= \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_{K-1} \neq \emptyset} \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_K \neq \emptyset} p_{123\cdots K} dZ_K dZ_2 \wedge dZ_3 \cdots \wedge dZ_{K-1} \\
&= \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_{K-1} \neq \emptyset} 2\pi\left(a_{123\cdots K-1} P_K + p_{123\cdots K-1} A_K\right) dZ_2 \wedge dZ_3 \cdots \wedge dZ_{K-1} \\
&= 2\pi P_K \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_{K-1} \neq \emptyset} a_{123\cdots K-} dZ_2 \wedge dZ_3 \cdots \wedge dZ_{K-1} \\
&\quad + 2\pi A_K \int_{Z_1 \cap Z_2 \cap Z_3 \cdots \cap Z_{K-1} \neq \emptyset} p_{123\cdots K-1} dZ_2 \wedge dZ_3 \cdots \wedge dZ_{K-1} \\
&= 2\pi P_K a_{K-1} + 2\pi A_K p_{K-1}
\end{aligned}
. \qquad \text{(A 20)}
$$

Then

$$
\begin{aligned}
p_K &= 2\pi P_K a_{K-1} + 2\pi A_K p_{K-1} \\
&= 2\pi P_K a_{K-1} + 2\pi A_K \left(2\pi P_{K-1} a_{K-2} + 2\pi A_{K-1} p_{K-2}\right) \\
&= 2\pi P_K a_{K-1} + 2\pi A_K \left\{2\pi P_{K-1} a_{K-2} + 2\pi A_{K-1}\left(2\pi P_{K-2} a_{K-3} + 2\pi A_{K-2} p_{K-3}\right)\right\} \\
&= 2\pi P_K a_{K-1} + (2\pi)^2 P_{K-1} A_K a_{K-2} + (2\pi)^3 P_{K-2} A_K A_{K-1} a_{K-3} + \cdots
\end{aligned}
. \qquad \text{(A 21)}
$$

Substituting equation A 20 we obtain

$$\begin{aligned} p_K &= 2\pi P_K a_{K-1} + (2\pi)^2 P_{K-1} A_K a_{K-2} + (2\pi)^3 P_{K-2} A_K A_{K-1} a_{K-3} + \cdots \\ &= (2\pi)^{K-1} P_K A_{K-1} A_{K-2} \cdots A_1 + (2\pi)^{K-1} A_K P_{K-1} A_{K-2} \cdots A_1 + \cdots \\ &= (2\pi)^{K-1} \left(P_K A_{K-1} A_{K-2} \cdots A_1 + A_K P_{K-1} A_{K-2} A_{K-2} \cdots A_1 + \cdots \right) \\ &= (2\pi)^{K-1} \sum_{i=1}^{K} P_i \prod_{j \neq i} A_j \end{aligned} \quad . \qquad \text{(A 22)}$$

Finally, from equation A 17 we have

$$\begin{aligned} m_K &= 2\pi a_{K-1} + P_K p_{K-1} + 2\pi A_K m_{K-1} \\ &= (2\pi)^{K-1} A_{K-1} A_{K-2} \cdots A_1 \\ &\quad + (2\pi)^{K-2} P_K \left(P_{K-1} A_{K-2} A_{K-3} \cdots A_1 + A_{K-1} P_{K-2} A_{K-3} \cdots A_1 + \cdots \right) \\ &\quad + 2\pi A_K m_{K-1} \\ &= (2\pi)^{K-1} \left(A_{K-1} A_{K-2} \cdots A_1 + A_K A_{K-2} A_{K-3} \cdots A_1 \right) \\ &\quad + (2\pi)^{K-2} \begin{pmatrix} P_K P_{K-1} A_{K-2} A_{K-3} \cdots A_1 + P_K A_{K-1} P_{K-2} A_{K-3} \cdots A_1 + \cdots \\ + A_K P_{K-1} P_{K-2} A_{K-3} A_{K-4} \cdots A_1 + A_K P_{K-1} A_{K-2} P_{K-3} A_{K-4} \cdots A_1 + \cdots \end{pmatrix} \\ &\quad + (2\pi)^2 A_K A_{K-1} m_{K-2} \\ &= (2\pi)^{K-1} \left(A_{K-1} A_{K-2} \cdots A_1 + A_K A_{K-2} A_{K-3} \cdots A_1 + A_K A_{K-1} A_{K-3} A_{K-4} \cdots A_1 \right) \\ &\quad + (2\pi)^{K-2} \begin{pmatrix} P_K P_{K-1} A_{K-2} A_{K-3} A_{K-4} \cdots A_1 + P_K A_{K-1} P_{K-2} A_{K-3} A_{K-4} \cdots A_1 + \cdots \\ + A_K P_{K-1} P_{K-2} A_{K-3} A_{K-4} \cdots A_1 + A_K P_{K-1} A_{K-2} P_{K-3} A_{K-4} \cdots A_1 + \cdots \\ + A_K A_{K-1} P_{K-2} P_{K-3} A_{K-4} \cdots A_1 + A_K A_{K-1} P_{K-2} A_{K-3} P_{K-4} \cdots A_1 + \cdots \end{pmatrix} . \\ &\quad + (2\pi)^3 A_K A_{K-1} A_{K-2} m_{K-3} \\ &= (2\pi)^{K-1} \sum_i \prod_{j \neq i} A_i + (2\pi)^{K-2} \sum_i \sum_{j \neq i} P_i P_j \prod_{k \neq i,j} A_k \end{aligned}$$

(A 23)

The measure of all sets of regions congruent to R_2, R_3, ..., R_K intersecting R_1 is written as

$$m\left(R_2, R_3 \cdots, R_K; R_1 \cap R_2 \cap \cdots \cap R_K \neq \varnothing\right) = (2\pi A)^{K-1}. \qquad \text{(A 24)}$$

Therefore,

$$\begin{aligned} \Pr\left(Z_1 \cap Z_2 \cap \cdots \cap Z_K \neq \varnothing\right) &= \frac{m_K}{m\left(R_2, R_3 \cdots, R_K; R_1 \cap R_2 \cap \cdots \cap R_K \neq \varnothing\right)} \\ &= \frac{(2\pi)^{K-1} \sum_i \prod_{j \neq i} A_i + (2\pi)^{K-2} \sum_i \sum_{j \neq i} P_i P_j \prod_{k \neq i,j} A_k}{(2\pi A)^{K-1}} . \\ &= \frac{2\pi \sum_i \prod_{j \neq i} A_i + \sum_i \sum_{j \neq i} P_i P_j \prod_{k \neq i,j} A_k}{2\pi A^{K-1}} \end{aligned}$$

(A 25)